Global well posedness and scattering for the elliptic and non-elliptic derivative nonlinear Schrödinger equations with small data

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Abstract

We study the Cauchy problem for the generalized elliptic and non-elliptic derivative nonlinear Schrödinger equations, the existence of the scattering operators and the global well posedness of solutions with small data in Besov spaces $B^s_{2,1}(\mathbb{R}^n)$ and in modulation spaces $M^s_{2,1}(\mathbb{R}^n)$ are obtained. In one spatial dimension, we get the sharp well posedness result with small data in critical homogeneous Besov spaces $\dot{B}^s_{2,1}$. As a by-product, the existence of the scattering operators with small data is also shown. In order to show these results, the global versions of the estimates for the maximal functions on the elliptic and non-elliptic Schrödinger groups are established.

Key words: Derivative nonlinear Schrödinger equation, elliptic and non-elliptic cases, estimates for the maximal function, global well posedness, small data.

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1 Introduction

We consider the Cauchy problem for the generalized derivative nonlinear Schrödinger equation (gNLS)

$$\mathrm{i}u_t + \Delta_{\pm} u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad u(0, x) = u_0(x), \tag{1.1}$$

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where u is a complex valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\Delta_{\pm} u = \sum_{i=1}^{n} \varepsilon_i \partial_{x_i}^2, \quad \varepsilon_i \in \{1, -1\}, \quad i = 1, ..., n, \tag{1.2}$$

$\nabla = (\partial_{x_1}, ..., \partial_{x_n})$, $F : \mathbb{C}^{2n+2} \to \mathbb{C}$ is a polynomial,

$$F(z) = P(z_1, ..., z_{2n+2}) = \sum_{m+1 \le |\beta| \le M+1} c_\beta z^\beta, \quad c_\beta \in \mathbb{C}, \tag{1.3}$$

$m, M \in \mathbb{N}$ will be given below.

There is a large literature which is devoted to the study of (1.1). Roughly speaking, three kinds of methods have been developed for the local and global well posedness of (1.1). The first one is the energy method, which is mainly useful to the elliptic case $\Delta_{\pm} = \Delta = \partial_{x_1}^2 + ... + \partial_{x_n}^2$, see Klainerman [21], Klainerman and Ponce [22], where the global classical solutions were obtained for the small Cauchy data with sufficient regularity and decay at infinity, F is assumed to satisfy an energy structure condition Re $\partial F / \partial(\nabla u) = 0$. Chihara [6,7] removed the condition Re $\partial F/\partial(\nabla u) = 0$ by using the smooth operators and the commutative estimates between the first order partial differential operators and $\mathrm{i}\partial_t + \Delta$, suitable decay conditions on the Cauchy data are still required in [6,7]. Recently, Ozawa and Zhang [25] removed the assumptions on the decay at infinity of the initial data. They obtained that if $n \ge 3$, $s > n/2+2$, $u_0 \in H^s$ is small enough, F is a smooth function vanishing of the third order at origin with Re $\partial F/\partial(\nabla u) = \nabla(\theta(|u|^2))$, $\theta \in C^2$, $\theta(0) = 0$, then (1.1) has a unique classical global solution $u \in (C_w \cap L^\infty)(\mathbb{R}, H^s) \cap C(\mathbb{R}, H^{s-1}) \cap L^2(\mathbb{R}; H^{s-1}_{2n/(n-2)})$. The main tools used in [25] are the gauge transform techniques, the energy method together with the endpoint Strichartz estimates.

The second way consists in using the $X^{s,b}$-like spaces, see Bourgain [3] and it has been developed by many authors (see [2,4,15] and references therein). This method depends on both the dispersive property of the linear equation and the structure of the nonlinearities, which is very useful for the lower regularity initial data.

The third method is to mainly use the dispersive smooth effects of the linear Schrödinger equation, see Kenig, Ponce and Vega [17,18]. The crucial point is that the Schrödinger group has the following locally smooth effects ($n \ge 2$):

$$\sup_{\alpha \in \mathbb{Z}^n} \|e^{\mathrm{i}t\Delta} u_0\|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \lesssim \|u_0\|_{\dot H^{-1/2}}, \tag{1.4}$$

$$\sup_{\alpha \in \mathbb{Z}^n} \left\| \nabla \int_0^t e^{\mathrm{i}(t-s)\Delta} f(s) ds \right\|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \|f\|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)}, \tag{1.5}$$

where Q_α is the unit cube with center at α. Estimate (1.5) contains one order smooth effect, which can be used to control the derivative terms in

the nonlinearities. Such smooth effect estimates are also adapted to the non-elliptic Schrödinger group, i.e., (1.4) and (1.5) still hold if we replace $e^{\mathrm{i}t\Delta}$ by $e^{\mathrm{i}t\Delta_\pm}$. Some earlier estimates related to (1.4) were due to Constantin and Saut [5], Sjölin [26] and Vega [34]. In [17,18], the local well posedness of (1.1) in both elliptic and non-elliptic cases was established for sufficiently smooth large Cauchy data ($m \geq 1$, $u_0 \in H^s$ with $s > n/2$ large enough). Moreover, they showed that the solutions are almost global if the initial data are sufficiently small, i.e., the maximal existing time of solutions tends to infinity as initial data tends to 0. Recently, the local well posedness results have been generalized to the quasi-linear (ultrahyperbolic) Schrödinger equations, see [19,20]. As far as the authors can see, the existence of the scattering operators for Eq. (1.1) and the global well posedness of (1.1) in the non-elliptic cases are unknown.

1.1 Main results

In this paper, we mainly apply the third method to study the global well posedness and the existence of the scattering operators of (1.1) in both the elliptic and non-elliptic cases with small data in $B^s_{2,1}$, $s > 3/2 + n/2$. We now state our main results, the notations used in this paper can be found in Sections 1.3 and 1.4.

Theorem 1.1 *Let $n \geq 2$ and $s > n/2 + 3/2$. Let $F(z)$ be as in* (1.3) *with $2 + 4/n \leq m \leq M < \infty$. We have the following results.*

(i) *If $\|u_0\|_{B^s_{2,1}} \leq \delta$ for $n \geq 3$, and $\|u_0\|_{B^s_{2,1}\cap \dot H^{-1/2}} \leq \delta$ for $n = 2$, where $\delta > 0$ is a suitably small number, then* (1.1) *has a unique global solution $u \in C(\mathbb{R},\ B^s_{2,1}) \cap X_0$, where*

$$X_0 = \left\{ u \ : \ \begin{array}{l} \|D^\beta u\|_{\ell^{1,s-1/2}_{\triangle}\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \lesssim \delta, \quad |\beta| \leq 1 \\ \|D^\beta u\|_{\ell^{1,s-1/2}_{\triangle}\ell^{2+4/n}_\alpha(L^\infty_{t,x}\cap(L^{2m}_t L^\infty_x)(\mathbb{R}\times Q_\alpha))} \lesssim \delta, \quad |\beta| \leq 1 \end{array} \right\}. \tag{1.6}$$

Moreover, for $n \geq 3$, the scattering operator of Eq. (1.1) *carries the ball $\{u : \|u\|_{B^s_{2,1}} \leq \delta\}$ into $B^s_{2,1}$.*

(ii) *If $s + 1/2 \in \mathbb{N}$ and $\|u_0\|_{H^s} \leq \delta$ for $n \geq 3$, and $\|u_0\|_{H^s\cap \dot H^{-1/2}} \leq \delta$ for $n = 2$, where $\delta > 0$ is a suitably small number, then* (1.1) *has a unique global solution $u \in C(\mathbb{R},\ H^s) \cap X$, where*

$$X = \left\{ u \ : \ \begin{array}{l} \|D^\beta u\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \lesssim \delta, \ |\beta| \leq s + 1/2 \\ \|D^\beta u\|_{\ell^{2+4/n}_\alpha(L^\infty_{t,x}\cap(L^{2m}_t L^\infty_x)(\mathbb{R}\times Q_\alpha))} \lesssim \delta, \ |\beta| \leq 1 \end{array} \right\}. \tag{1.7}$$

Moreover, for $n \geq 3$, the scattering operator of Eq. (1.1) *carries the ball* $\{u : \|u\|_{H^s} \leq \delta\}$ *into* H^s.

We now illustrate the proof of (ii) in Theorem 1.1. Let us consider the equivalent integral equation

$$u(t) = S(t)u_0 - \mathrm{i}\mathscr{A} F(u, \bar{u}, \nabla u, \nabla \bar{u}), \tag{1.8}$$

where

$$S(t) := e^{\mathrm{i}t\Delta_\pm}, \quad \mathscr{A} f := \int_0^t e^{\mathrm{i}(t-s)\Delta_\pm} f(s) ds. \tag{1.9}$$

If one applies the local smooth effect estimate (1.5) to control the derivative terms in the nonlinearities, then the working space should contains the space $\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))$. For simplicity, we consider the case $F(u, \bar{u}, \nabla u, \nabla \bar{u}) = (\partial_{x_1} u)^{\nu+1}$. By (1.4) and (1.5), we immediately have

$$\begin{aligned} \|\nabla u\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} &\lesssim \|u_0\|_{H^{1/2}} + \sum_{\alpha \in \mathbb{Z}^n} \|(\partial_{x_1} u)^{\nu+1}\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \\ &\lesssim \|u_0\|_{H^{1/2}} + \|\nabla u\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \|\nabla u\|^\nu_{\ell^\nu_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))}. \end{aligned} \tag{1.10}$$

Hence, one needs to control $\|\nabla u\|_{\ell^\nu_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))}$. In [17,18], it was shown that for $\nu = 2$,

$$\|S(t)u_0\|_{\ell^2_\alpha(L^\infty_{t,x}([0,T]\times Q_\alpha))} \leq C(T)\|u_0\|_{H^s}, \quad s > n/2 + 2. \tag{1.11}$$

In the elliptic case (1.11) holds for $s > n/2$. (1.11) is a time-local version which prevents us to get the global existence of solutions. So, it is natural to ask if there is a time-global version for the estimates of the maximal function. We can get the following

$$\|S(t)u_0\|_{\ell^\nu_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))} \leq C\|u_0\|_{H^s}, \quad s > n/2, \quad \nu \geq 2 + 4/n. \tag{1.12}$$

Applying (1.12), we have for any $s > n/2$,

$$\|\nabla u\|_{\ell^\nu_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))} \lesssim \|\nabla u_0\|_{H^s} + \|\nabla(\partial_{x_1} u)^{1+\nu}\|_{L^1(\mathbb{R}, H^s(\mathbb{R}^n))}. \tag{1.13}$$

One can get, say for $s = [n/2] + 1$,

$$\|\nabla(\partial_{x_1} u)^{1+\nu}\|_{L^1(\mathbb{R}, H^s(\mathbb{R}^n))} \lesssim \sum_{|\beta| \leq s+2} \|D^\beta u\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \|\nabla u\|^\nu_{\ell^\nu_\alpha(L^{2\nu}_t L^\infty_x(\mathbb{R}\times Q_\alpha))}. \tag{1.14}$$

Hence, we need to further estimate $\|D^\beta u\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}$ for all $|\beta| \leq s + 2$ and $\|\nabla u\|_{\ell^\nu_\alpha(L^{2\nu}_t L^\infty_x(\mathbb{R}\times Q_\alpha))}$. We can conjecture that a similar estimate to (1.10)

holds:

$$\sum_{|\beta|\le s+2}\|D^\beta u\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}$$
$$\lesssim \|u_0\|_{H^{s+3/2}}+\sum_{|\beta|\le s+2}\|D^\beta u\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}\|\nabla u\|^\nu_{\ell^\nu_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))}. \tag{1.15}$$

Finally, for the estimate of $\|\nabla u\|_{\ell^\nu_\alpha(L^{2\nu}_t L^\infty_x(\mathbb{R}\times Q_\alpha))}$, one needs the following

$$\|S(t)u_0\|_{\ell^\nu_\alpha(L^{2\nu}_t L^\infty_x(\mathbb{R}\times Q_\alpha))}\le C\|u_0\|_{H^{s-1/\nu}},\quad s>n/2,\ \ \nu\ge 2+4/n. \tag{1.16}$$

Using (1.16), the estimate of $\|\nabla u\|_{\ell^\nu_\alpha(L^{2\nu}_t L^\infty_x(\mathbb{R}\times Q_\alpha))}$ becomes easier than that of $\|\nabla u\|_{\ell^\nu_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))}$. Hence, the solution has a self-contained behavior by using the spaces $\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))$, $\ell^\nu_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))$ and $\ell^\nu_\alpha(L^{2\nu}_t L^\infty_x(\mathbb{R}\times Q_\alpha))$. We will give the details of the estimates (1.12) and (1.16) in Section 2. The nonlinear mapping estimates as in (1.14) and (1.15) will be given in Section 4.

Next, we use the frequency-uniform decomposition method developed in [31,32,33] to consider the case of initial data in modulation spaces $M^s_{2,1}$, which is the low regularity version of Besov spaces $B^{n/2+s}_{2,1}$, i.e., $B^{n/2+s}_{2,1}\subset M^s_{2,1}$ is a sharp embedding and $M^s_{2,1}$ has only s-order derivative regularity (see [27,29,32], for the final result, see [33]). We have the following local well posedness result with small rough initial data:

Theorem 1.2 *Let $n\ge 2$. Let $F(z)$ be as in* (1.3) *with $2\le m\le M<\infty$. Assume that $\|u_0\|_{M^2_{2,1}}\le\delta$ for $n\ge 3$, and $\|u_0\|_{M^2_{2,1}\cap\dot H^{-1/2}}\le\delta$ for $n=2$, where $\delta>0$ is sufficiently small. Then there exists a $T:=T(\delta)>0$ such that* (1.1) *has a unique local solution $u\in C([0,T],\ M^2_{2,1})\cap Y$, where*

$$Y=\left\{u\ :\ \begin{array}{l}\|D^\beta u\|_{\ell^{1,3/2}_\square\ell^\infty_\alpha(L^2_{t,x}([0,T]\times Q_\alpha))}\lesssim\delta,\ |\beta|\le 1\\ \|D^\beta u\|_{\ell^1_\square\ell^2_\alpha(L^\infty_{t,x}([0,T]\times Q_\alpha))}\lesssim\delta,\ |\beta|\le 1\end{array}\right\}. \tag{1.17}$$

Moreover, $\lim_{\delta\searrow 0}T(\delta)=\infty$.

The following is a global well posedness result with Cauchy data in modulation spaces $M^s_{2,1}$:

Theorem 1.3 *Let $n\ge 2$. Let $F(z)$ be as in* (1.3) *with $2+4/n\le m\le M<\infty$. Let $s>3/2+(n+2)/m$. Assume that $\|u_0\|_{M^s_{2,1}}\le\delta$ for $n\ge 3$, and $\|u_0\|_{M^s_{2,1}\cap\dot H^{-1/2}}\le\delta$ for $n=2$, where $\delta>0$ is a suitably small number. Then* (1.1) *has a unique global solution $u\in C(\mathbb{R},\ M^s_{2,1})\cap Z$, where*

$$Z=\left\{u\ :\ \begin{array}{l}\|D^\beta u\|_{\ell^{1,s-1/2}_\square\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}\lesssim\delta,\ |\beta|\le 1\\ \|D^\beta u\|_{\ell^1_\square\ell^m_\alpha(L^\infty_{t,x}\cap(L^{2m}_t L^\infty_x)(\mathbb{R}\times Q_\alpha))}\lesssim\delta,\ |\beta|\le 1\end{array}\right\}. \tag{1.18}$$

Moreover, for $n \geq 3$, the scattering operator of Eq. (1.1) *carries the ball* $\{u : \|u\|_{M^s_{2,1}} \leq \delta\}$ *into* $M^s_{2,1}$.

Finally, we consider one spatial dimension case. Denote

$$s_\kappa = \frac{1}{2} - \frac{2}{\kappa}, \quad \tilde{s}_\nu = \frac{1}{2} - \frac{1}{\nu}. \tag{1.19}$$

Theorem 1.4 *Let $n = 1$, $M \geq m \geq 4$, $u_0 \in \dot{B}^{1+\tilde{s}_M}_{2,1} \cap \dot{B}^{s_m}_{2,1}$. Assume that there exists a small $\delta > 0$ such that $\|u_0\|_{\dot{B}^{1+\tilde{s}_M}_{2,1} \cap \dot{B}^{s_m}_{2,1}} \leq \delta$. Then* (1.1) *has a unique global solution $u \in X = \{u \in \mathscr{S}'(\mathbb{R}^{1+1}) : \|u\|_X \lesssim \delta\}$, where*

$$\begin{aligned}
&\|u\|_X = \sup_{s_m \leq s \leq \tilde{s}_M} \sum_{i=0,1} \sum_{j\in\mathbb{Z}} \|\partial^i_x \triangle_j u\|_s \quad for \quad m > 4, \\
&\|u\|_X = \sum_{i=0,1} \Big(\|\partial^i_x u\|_{L^\infty_t L^2_x \cap L^6_{x,t}} + \sup_{\tilde{s}_m \leq s \leq \tilde{s}_M} \sum_{j\in\mathbb{Z}} \|\partial^i_x \triangle_j u\|_s \Big) \quad for \quad m = 4, \\
&\|\triangle_j v\|_s := 2^{sj} (\|\triangle_j v\|_{L^\infty_t L^2_x \cap L^6_{x,t}} + 2^{j/2} \|\triangle_j v\|_{L^\infty_x L^2_t}) \\
&\qquad\qquad + 2^{(s-\tilde{s}_m)j} \|\triangle_j v\|_{L^m_x L^\infty_t} + 2^{(s-\tilde{s}_M)j} \|\triangle_j v\|_{L^M_x L^\infty_t}.
\end{aligned} \tag{1.20}$$

Recall that the norm on homogeneous Besov spaces $\dot{B}^s_{2,1}$ can be defined in the following way:

$$\|f\|_{\dot{B}^s_{2,1}} = \sum_{j=-\infty}^{\infty} 2^{sj} \left(\int_{2^j}^{2^{j+1}} |\mathscr{F} f(\xi)|^2 d\xi \right)^{1/2}. \tag{1.21}$$

1.2 Remarks on main results

It seems that the regularity assumptions on initial data are not optimal in Theorems 1.1–1.3, but Theorem 1.4 presents the sharp regularity condition to the initial data. To illustrate the relation between the regularity index and the nonlinear power, we consider a simple cases of (1.1):

$$\mathrm{i}u_t + \Delta_\pm u = u^\nu_{x_1}, \quad u(0) = \phi. \tag{1.22}$$

Eq. (1.22) is invariant under the scaling $u \to u_\lambda = \lambda^{(2-\nu)/(\nu-1)} u(\lambda^2 t, \lambda x)$ and moreover,

$$\|\phi\|_{\dot{H}^s(\mathbb{R}^n)} = \|u_\lambda(0, \cdot)\|_{\dot{H}^s(\mathbb{R}^n)}, \quad s = 1 + \tilde{s}_{\nu-1} := 1 + n/2 - 1/(\nu - 1). \tag{1.23}$$

From this point of view, we say that $s = 1+\tilde{s}_{\nu-1}$ is the critical regularity index of (1.22). In [23], Molinet and Ribuad showed that (1.22) is ill-posed in one spatial dimension in the sense if $s_1 \neq \tilde{s}_{\nu-1}+1$, the flow map of equation (1.22)

$\phi \to u$ (if it exists) is not of class C^ν from $\dot B^{s_1}_{2,1}(\mathbb{R})$ to $C([0,\infty), \dot B^{s_1}_{2,1}(\mathbb{R}))$ at the origin $\phi = 0$. For each term in the polynomial nonlinearity $F(u, \bar u, \nabla u, \nabla \bar u)$ as in (1.3), we easily see that the critical index s can take any critical index between s_m and $1 + \tilde s_M$. So, our Theorem 1.4 give sharp result in the case $m \geq 4$. On the other hand, Christ [9] showed that in the case $\nu = 2$, $n = 1$, for any $s \in \mathbb{R}$, there exist initial data in H^s with arbitrarily small norm, for which the solution attains arbitrarily large norm after an arbitrarily short time (see also [24]). From Christ's result together with Theorems 1.4, we can expect that there exists $m_0 > 1$ (might be non-integer) so that for $\nu - 1 \geq m_0$, $s = 1 + \tilde s_{\nu-1}$ is the minimal regularity index to guarantee the well posedness of (1.22), at least for the local solutions and small data global solutions in H^s. However, it is not clear for us how to find the exact value of m_0 even in one spatial dimension.

However, in higher spatial dimensions, it seems that $1/2+1/M$-order derivative regularity is lost in Theorem 1.1 and we do not know how to attain the regularity index $s \geq 1 + \tilde s_M$.

In two dimensional case, if $\Delta_\pm = \Delta$ and the initial value u_0 is a radial function, we can remove the condition $u_0 \in \dot H^{-1/2}$, $\|u_0\|_{\dot H^{-1/2}} \leq \delta$ by using the endpoint Strichartz estimates as in the case $n \geq 3$.

Considering the nonlinearity $F(u, \nabla u) = (1 - |u|^2)^{-1} |\nabla u|^{2k} u$, Theorem 1.2 holds for the case $k \geq 1$. Theorems 1.1 and 1.3 hold for the case $k \geq 2$. Since $(1 - |u|^2)^{-1} = \sum_{k=0}^\infty |u|^{2k}$, one easily sees that we can use the same way as in the proof of our main results to handle this kind of nonlinearity.

1.3 Notations

Throughout this paper, we will always use the following notations. $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ stand for the Schwartz space and its dual space, respectively. We denote by $L^p(\mathbb{R}^n)$ the Lebesgue space, $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R}^n)}$. The Bessel potential space is defined by $H^s_p(\mathbb{R}^n) := (I - \Delta)^{-s/2} L^p(\mathbb{R}^n)$, $H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$, $\dot H^s(\mathbb{R}^n) = (-\Delta)^{-s/2} L^2(\mathbb{R}^n)$.[1] For any quasi-Banach space X, we denote by X^* its dual space, by $L^p(I, X)$ the Lebesgue-Bochner space, $\|f\|_{L^p(I,X)} := (\int_I \|f(t)\|^p_X dt)^{1/p}$. If $X = L^r(\Omega)$, then we write $L^p(I, L^r(\Omega)) = L^p_t L^r_x (I \times \Omega)$ and $L^p_{t,x}(I \times \Omega) = L^p_t L^p_x(I \times \Omega)$. Let Q_α be the unit cube with center at $\alpha \in \mathbb{Z}^n$, i.e., $Q_\alpha = \alpha + Q_0, Q_0 = \{x = (x_1, ... x_n) : -1/2 \leq x_i < 1/2\}$. We also needs

[1] $\mathbb{R}^n$ will be omitted in the definitions of various function spaces if there is no confusion.

the function spaces $\ell^q_\alpha(L^p_t L^r_x(I\times Q_\alpha))$,

$$\|f\|_{\ell^q_\alpha(L^p_t L^r_x(I\times Q_\alpha))} := \left(\sum_{\alpha\in\mathbb{Z}^n} \|f\|^q_{L^p_t L^r_x(I\times Q_\alpha)}\right)^{1/q}.$$

We denote by $\mathscr{F}$ ($\mathscr{F}^{-1}$) the (inverse) Fourier transform for the spatial variables; by $\mathscr{F}_t$ ($\mathscr{F}_t^{-1}$) the (inverse) Fourier transform for the time variable and by $\mathscr{F}_{t,x}$ ($\mathscr{F}_{t,x}^{-1}$) the (inverse) Fourier transform for both time and spatial variables, respectively. If there is no explanation, we always denote by $\varphi_k(\cdot)$ the dyadic decomposition functions as in (1.25); and by $\sigma_k(\cdot)$ the uniform decomposition functions as in (1.27). $u\star v$ and $u*v$ will stand for the convolution on time and on spatial variables, respectively, i.e.,

$$(u\star v)(t,x)=\int_{\mathbb{R}} u(t-\tau,x)v(\tau,x)d\tau, \quad (u*v)(t,x)=\int_{\mathbb{R}^n} u(t,x-y)v(t,y)dy.$$

$\mathbb{R},\mathbb{N}$ and $\mathbb{Z}$ will stand for the sets of reals, positive integers and integers, respectively. $c<1$, $C>1$ will denote positive universal constants, which can be different at different places. $a\lesssim b$ stands for $a\le Cb$ for some constant $C>1$, $a\sim b$ means that $a\lesssim b$ and $b\lesssim a$. We denote by p' the dual number of $p\in[1,\infty]$, i.e., $1/p+1/p'=1$. For any $a>0$, we denote by $[a]$ the minimal integer that is larger than or equals to a. $B(x,R)$ will denote the ball in $\mathbb{R}^n$ with center x and radial R.

1.4 Besov and modulation spaces

Let us recall that Besov spaces $B^s_{p,q}:=B^s_{p,q}(\mathbb{R}^n)$ are defined as follows (cf. [1,30]). Let $\psi:\mathbb{R}^n\to[0,1]$ be a smooth radial bump function adapted to the ball $B(0,2)$:

$$\psi(\xi)=\begin{cases}1, & |\xi|\le 1,\\ \text{smooth}, & |\xi|\in[1,2],\\ 0, & |\xi|\ge 2.\end{cases} \tag{1.24}$$

We write $\delta(\cdot):=\psi(\cdot)-\psi(2\,\cdot)$ and

$$\varphi_j:=\delta(2^{-j}\cdot) \ \text{ for } \ j\ge 1; \quad \varphi_0:=1-\sum_{j\ge1}\varphi_j. \tag{1.25}$$

We say that $\triangle_j:=\mathscr{F}^{-1}\varphi_j\mathscr{F}$, $\quad j\in\mathbb{N}\cup\{0\}$ are the dyadic decomposition

operators. Beove spaces $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n)$ are defined in the following way:

$$B^s_{p,q} = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \ \|f\|_{B^s_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{sjq} \| \triangle_j f\|_p^q \right)^{1/q} < \infty \right\}. \tag{1.26}$$

Now we recall the definition of modulation spaces (see [12,13,31,32,33]). Here we adopt an equivalent norm by using the uniform decomposition to the frequency space. Let $\rho \in \mathscr{S}(\mathbb{R}^n)$ and $\rho : \mathbb{R}^n \to [0,1]$ be a smooth radial bump function adapted to the ball $B(0, \sqrt{n})$, say $\rho(\xi) = 1$ as $|\xi| \le \sqrt{n}/2$, and $\rho(\xi) = 0$ as $|\xi| \ge \sqrt{n}$. Let ρ_k be a translation of ρ: $\rho_k(\xi) = \rho(\xi - k), \ k \in \mathbb{Z}^n$. We write

$$\sigma_k(\xi) = \rho_k(\xi) \left(\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n. \tag{1.27}$$

Denote

$$\Box_k := \mathscr{F}^{-1} \sigma_k \mathscr{F}, \quad k \in \mathbb{Z}^n, \tag{1.28}$$

which are said to be the frequency-uniform decomposition operators. For any $k \in \mathbb{Z}^n$, we write $\langle k \rangle = \sqrt{1 + |k|^2}$. Let $s \in \mathbb{R}$, $0 < p, q \le \infty$. Modulation spaces $M^s_{p,q} = M^s_{p,q}(\mathbb{R}^n)$ are defined as:

$$M^s_{p,q} = \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \ \|f\|_{M^s_{p,q}} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \| \Box_k f\|_p^q \right)^{1/q} < \infty \right\}. \tag{1.29}$$

We will use the function space $\ell^{1,s}_{\Box} \ell^q_\alpha (L^p_t L^r_x (I \times Q_\alpha))$ which contains all of the functions $f(t,x)$ so that the following norm is finite:

$$\|f\|_{\ell^{1,s}_{\Box} \ell^q_\alpha (L^p_t L^r_x (I \times Q_\alpha))} := \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \left(\sum_{\alpha \in \mathbb{Z}^n} \|\Box_k f\|^q_{L^p_t L^r_x (I \times Q_\alpha)} \right)^{1/q}. \tag{1.30}$$

Similarly, we can define the space $\ell^{1,s}_{\triangle} \ell^q_\alpha (L^p_{t,x} (I \times Q_\alpha))$ with the following norm:

$$\|f\|_{\ell^{1,s}_{\triangle} \ell^q_\alpha (L^p_t L^r_x (I \times Q_\alpha))} := \sum_{j=0}^{\infty} 2^{sj} \left(\sum_{\alpha \in \mathbb{Z}^n} \|\triangle_j f\|^q_{L^p_t L^r_x (I \times Q_\alpha)} \right)^{1/q}. \tag{1.31}$$

A special case is $s = 0$, we write $\ell^{1,0}_{\Box} \ell^q_\alpha (L^p_t L^r_x (I \times Q_\alpha)) = \ell^1_{\Box} \ell^q_\alpha (L^p_t L^r_x (I \times Q_\alpha))$ and $\ell^{1,0}_{\triangle} \ell^q_\alpha (L^p_t L^r_x (I \times Q_\alpha)) = \ell^1_{\triangle} \ell^q_\alpha (L^p_t L^r_x (I \times Q_\alpha))$.

The rest of this paper is organized as follows. In Section 2 we give the details of the estimates for the maximal function in certain function spaces. Section 3 is devoted to considering the spatial local versions for the Strichartz

estimates and giving some remarks on the estimates of the local smooth effects. In Sections 4–7 we prove our main Theorems 1.1–1.4, respectively.

2 Estimates for the maximal function

2.1 Time-local version

Recall that $S(t) = e^{-\mathrm{i}t\triangle_\pm} = \mathscr{F}^{-1}e^{\mathrm{i}t|\xi|^2_\pm}\mathscr{F}$, where

$$|\xi|^2_\pm = \sum_{j=1}^n \varepsilon_j \xi_j^2, \quad \varepsilon_j = \pm 1. \tag{2.1}$$

Kenig, Ponce and Vega [17] showed the following maximal function estimate:

$$\left(\sum_{\alpha\in\mathbb{Z}^n} \|S(t)u_0\|^2_{L^\infty_{t,x}([0,T]\times Q_\alpha)}\right)^{1/2} \lesssim C(T)\|u_0\|_{H^s}, \tag{2.2}$$

where $s \geq 2 + n/2$. If $S(t) = e^{-\mathrm{i}t\triangle}$, then (2.2) holds for $s > n/2$, $C(T) = (1+T)^s$. Using the frequency-uniform decomposition method, we can get the following

Proposition 2.1 *There exists a constant $C(T) > 1$ which depends only on T and n such that*

$$\sum_{k\in\mathbb{Z}^n}\left(\sum_{\alpha\in\mathbb{Z}^n} \|\,\Box_k S(t)u_0\|^2_{L^\infty_{t,x}([0,T]\times Q_\alpha)}\right)^{1/2} \leq C(T)\|u_0\|_{M^{1/2}_{2,1}}, \tag{2.3}$$

In particular, for any $s > (n+1)/2$,

$$\left(\sum_{\alpha\in\mathbb{Z}^n} \|\,S(t)u_0\|^2_{L^\infty_{t,x}([0,T]\times Q_\alpha)}\right)^{1/2} \leq C(T)\|u_0\|_{H^s}. \tag{2.4}$$

Proof. By the duality, it suffices to prove that

$$\int_0^T (S(t)u_0, \psi(t))dt \lesssim \|u_0\|_{M^{1/2}_{2,1}} \sup_{k\in\mathbb{Z}^n}\left(\sum_{\alpha\in\mathbb{Z}^n} \|\Box_k\psi(t)\|^2_{L^1_{t,x}([0,T]\times Q_\alpha)}\right)^{1/2}. \tag{2.5}$$

Since $(M^{1/2}_{2,1})^* = M^{-1/2}_{2,\infty}$, we have

$$\int_0^T (S(t)u_0, \psi(t))dt \le \|u_0\|_{M^{1/2}_{2,1}} \left\| \int_0^T S(-t)\psi(t)dt \right\|_{M^{-1/2}_{2,\infty}}. \tag{2.6}$$

Recalling that $\|f\|_{M^{-1/2}_{2,\infty}} = \sup_{k\in\mathbb{Z}^n} \langle k\rangle^{-1/2} \|\Box_k f\|_2$, we need to estimate

$$\begin{aligned}
&\left\| \Box_k \int_0^T S(-t)\psi(t)dt \right\|_2^2 \\
&= \int_0^T \left(\Box_k \psi(t),\ \int_0^T S(t-\tau)\Box_k \psi(\tau)d\tau \right) dt \\
&\le \sum_{\alpha\in\mathbb{Z}^n} \|\Box_k\psi\|_{L^1_{t,x}([0,T]\times Q_\alpha)} \left\| \int_0^T S(t-\tau)\Box_k\psi(\tau)d\tau \right\|_{L^\infty_{t,x}([0,T]\times Q_\alpha)} \\
&\le \|\Box_k\psi\|_{\ell^2_\alpha(L^1_{t,x}([0,T]\times Q_\alpha))} \left\| \int_0^T S(t-\tau)\Box_k\psi(\tau)d\tau \right\|_{\ell^2_\alpha(L^\infty_{t,x}([0,T]\times Q_\alpha))}.
\end{aligned} \tag{2.7}$$

If one can show that

$$\left\| \int_0^T S(t-\tau)\Box_k\psi(\tau)d\tau \right\|_{\ell^2_\alpha(L^\infty_{t,x}([0,T]\times Q_\alpha))} \lesssim C(T)\langle k\rangle \|\Box_k\psi\|_{\ell^2_\alpha(L^1_{t,x}([0,T]\times Q_\alpha))}, \tag{2.8}$$

then from (2.6)–(2.8) we obtain that (2.5) holds. Denote

$$\Lambda := \{\ell\in\mathbb{Z}^n : \operatorname{supp}\sigma_\ell \cap \operatorname{supp}\sigma_0 \ne \varnothing\}. \tag{2.9}$$

In the following we show (2.8). In view of Young's inequality, we have

$$\begin{aligned}
&\left\| \int_0^T S(t-\tau)\Box_k\psi(\tau)d\tau \right\|_{L^\infty_{t,x}([0,T]\times Q_\alpha)} \\
&\lesssim \sum_{\ell\in\Lambda} \left\| \int_0^T S(t-\tau)\Box_{k+\ell}\Box_k\psi(\tau)d\tau \right\|_{L^\infty_{t,x}([0,T]\times Q_\alpha)} \\
&= \sum_{\ell\in\Lambda} \left\| \int_0^T [\mathscr{F}^{-1}(e^{\mathrm{i}(t-\tau)|\xi|^2_\pm}\sigma_{k+\ell})] * \Box_k\psi(\tau)d\tau \right\|_{L^\infty_{t,x}([0,T]\times Q_\alpha)} \\
&\le \sum_{\ell\in\Lambda}\sum_{\beta\in\mathbb{Z}^n} \left\| \mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_{k+\ell}) \right\|_{L^\infty_{t,x}([-T,T]\times Q_\beta)} \|\Box_k\psi\|_{L^1_{t,x}([0,T]\times(Q_\alpha - Q_\beta))}.
\end{aligned} \tag{2.10}$$

From (2.10) and Minkowski's inequality that

$$\begin{aligned}
&\left\| \int_0^T S(t-\tau)\Box_k\psi(\tau)d\tau \right\|_{\ell^2_\alpha(L^\infty_{t,x}([0,T]\times Q_\alpha))} \\
&\le \sum_{\ell\in\Lambda}\sum_{\beta\in\mathbb{Z}^n} \left\| \mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_{k+\ell}) \right\|_{L^\infty_{t,x}([-T,T]\times Q_\beta)} \|\Box_k\psi\|_{\ell^2_\alpha(L^1_{t,x}([0,T]\times(Q_\alpha - Q_\beta)))}.
\end{aligned} \tag{2.11}$$

It is easy to see that

$$\|\Box_k \psi\|_{\ell^2_\alpha(L^1_{t,x}([0,T]\times(Q_\alpha - Q_\beta)))} \lesssim \|\Box_k \psi\|_{\ell^2_\alpha(L^1_{t,x}([0,T]\times Q_\alpha))}. \tag{2.12}$$

Hence, in order to prove (2.8), it suffices to prove that

$$\sum_{\beta\in\mathbb{Z}^n} \left\|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_k)\right\|_{L^\infty_{t,x}([0,T]\times Q_\beta)} \lesssim C(T)\langle k\rangle. \tag{2.13}$$

In fact, observing the following identity,

$$|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_k)| = |\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)(\cdot + 2tk_\pm)|, \tag{2.14}$$

where $k_\pm = (\varepsilon_1 k_1, ..., \varepsilon_n k_n)$, we have

$$\|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_k)\|_{L^\infty_{t,x}([0,T]\times Q_\beta)} \le \|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)\|_{L^\infty_{t,x}([0,T]\times Q^*_{\beta,k})}, \tag{2.15}$$

where

$$Q^*_{\beta,k} = \{x:\ x\in 2tk_\pm + Q_\beta \ \text{ for some } \ t\in[0,T]\}.$$

Denote $\Lambda_{\beta,k} = \{\beta':\ Q_{\beta'}\cap Q^*_{\beta,k} \neq \varnothing\}$. It follows from (2.15) that

$$\sum_{\beta\in\mathbb{Z}^n} \|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_k)\|_{L^\infty_{t,x}([0,T]\times Q_\beta)} \le \sum_{\beta\in\mathbb{Z}^n}\sum_{\beta'\in\Lambda_{\beta,k}} \|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)\|_{L^\infty_{t,x}([0,T]\times Q_{\beta'})}. \tag{2.16}$$

Since each $E_{\beta,k}$ overlaps at most $O(T\langle k\rangle)$ many $Q_{\beta'}$, $\beta'\in\mathbb{Z}^n$, one can easily verify that in the sums of the right hand side of (2.16), each $\|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)\|_{L^\infty_{t,x}([0,T]\times Q_{\beta'})}$ repeats at most $O(T\langle k\rangle)$ times. Hence, we have

$$\sum_{\beta\in\mathbb{Z}^n} \|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_k)\|_{L^\infty_{t,x}([0,T]\times Q_\beta)} \lesssim \langle k\rangle \sum_{\beta\in\mathbb{Z}^n} \|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)\|_{L^\infty_{t,x}([0,T]\times Q_\beta)}. \tag{2.17}$$

Finally, it suffices to show that

$$\sum_{\beta\in\mathbb{Z}^n} \|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)\|_{L^\infty_{t,x}([0,T]\times Q_\beta)} \le C(T). \tag{2.18}$$

Denote $\nabla_{t,x} = (\partial_t, \partial_{x_1}, ..., \partial_{x_n})$. By the Sobolev inequality,

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}^n} \|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)\|_{L^\infty_{t,x}([0,T]\times Q_\beta)} \lesssim& \sum_{\beta\in\mathbb{Z}^n} \|\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)\|_{L^{2n}_{t,x}([0,T]\times Q_\beta)} \\ &+ \sum_{\beta\in\mathbb{Z}^n} \|\nabla_{t,x}\mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm}\sigma_0)\|_{L^{2n}_{t,x}([0,T]\times Q_\beta)} \\ =& I + II. \end{aligned} \tag{2.19}$$

By Hölder's inequality, we have

$$
\begin{aligned}
II &\lesssim \left(\sum_{\beta \in \mathbb{Z}^n} \left\| (1+|x|^2)^n \nabla_{t,x} \mathscr{F}^{-1}(e^{\mathrm{i}t|\xi|^2_\pm} \sigma_0) \right\|^{2n}_{L^{2n}_{t,x}([0,T]\times Q_\beta)} \right)^{1/2n} \\
&\lesssim \sum_{i=1}^n \left\| \mathscr{F}^{-1}(I-\Delta)^n (e^{\mathrm{i}t|\xi|^2_\pm} \xi_i \sigma_0) \right\|_{L^{2n}_{t,x}([0,T]\times \mathbb{R}^n)} \\
&\quad + \left\| \mathscr{F}^{-1}(I-\Delta)^n (e^{\mathrm{i}t|\xi|^2_\pm} |\xi|^2_\pm \sigma_0) \right\|_{L^{2n}_{t,x}([0,T]\times \mathbb{R}^n)} \\
&\lesssim C(T).
\end{aligned} \tag{2.20}
$$

One easily sees that I has the same bound as that of II. The proof of (2.3) is finished. Noticing that $H^s \subset M^{1/2}_{2,1}$ if $s > (n+1)/2$ (cf. [29,32,33]), we immediately have (2.4). □

2.2 Time-global version

Recall that we have the following equivalent norm on Besov spaces ([1,30]):

Lemma 2.2 *Let $1 \le p, q \le \infty$, $\sigma > 0$, $\sigma \notin \mathbb{N}$. Then we have*

$$
\|f\|_{B^\sigma_{p,q}} \sim \sum_{|\beta| \le [\sigma]} \|D^\beta f\|_{L^p(\mathbb{R}^n)} + \sum_{|\beta| \le [\sigma]} \left(\int_{\mathbb{R}^n} |h|^{-n-q\{\sigma\}} \| \bigtriangleup_h D^\beta f\|^q_{L^p(\mathbb{R}^n)} dh \right)^{1/q}, \tag{2.21}
$$

where $\bigtriangleup_h f = f(\cdot + h) - f(\cdot)$, $[\sigma]$ denotes the minimal integer that is larger than or equals to σ, $\{\sigma\} = \sigma - [\sigma]$.

Taking $p = q$ in Lemma 2.2, one has that

$$
\|f\|^p_{B^\sigma_{p,p}} \sim \sum_{|\beta| \le [\sigma]} \|D^\beta f\|^p_{L^p(\mathbb{R}^n)} + \sum_{|\beta| \le [\sigma]} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{| \bigtriangleup_h D^\beta f(x)|^p}{|h|^{n+p\{\sigma\}}} dx dh. \tag{2.22}
$$

Lemma 2.3 *Let $1 < p < \infty$, $s > 1/p$. Then we have*

$$
\left(\sum_{\alpha \in \mathbb{Z}^n} \|u\|^p_{L^\infty_{t,x}(\mathbb{R}\times Q_\alpha)} \right)^{1/p} \lesssim \|(I - \partial_t^2)^{s/2} u\|_{L^p(\mathbb{R}, B^{ns}_{p,p}(\mathbb{R}^n))}. \tag{2.23}
$$

Proof. We divide the proof into the following two cases.

Case 1. $ns \notin \mathbb{N}$. Due to $H^s_p(\mathbb{R}) \subset L^\infty(\mathbb{R})$, we have

$$
\|u\|_{L^\infty_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \|(I - \partial_t^2)^{s/2} u\|_{L^\infty_x L^p_t(Q_\alpha \times \mathbb{R})}
$$

$$\leq \|(I-\partial_t^2)^{s/2}u\|_{L_t^pL_x^\infty(\mathbb{R}\times Q_\alpha)}. \tag{2.24}$$

Recalling that $\sigma_\alpha(x) \gtrsim 1$ for all $x \in Q_\alpha$ and $\alpha \in \mathbb{Z}^n$, we have from (2.24) that

$$\|u\|_{L^\infty_{t,x}(\mathbb{R}\times Q_\alpha)} \leq \|(I-\partial_t^2)^{s/2}\sigma_\alpha u\|_{L_t^pL_x^\infty(\mathbb{R}\times \mathbb{R}^n)}. \tag{2.25}$$

Since $B^{ns}_{p,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, in view of (2.25), one has that

$$\|u\|_{L^\infty_{t,x}(\mathbb{R}\times Q_\alpha)} \leq \|(I-\partial_t^2)^{s/2}\sigma_\alpha u\|_{L^p(\mathbb{R},B^{ns}_{p,p}(\mathbb{R}^n))}. \tag{2.26}$$

For simplicity, we denote $v=(I-\partial_t^2)^{s/2}u$. By (2.22) and (2.26) we have

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^n}\|u\|^p_{L^\infty_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim &\sum_{|\beta|\leq[ns]}\sum_{\alpha\in\mathbb{Z}^n}\int_{\mathbb{R}}\|D^\beta(\sigma_\alpha v)(t)\|^p_{L^p(\tilde{Q}_\alpha)}dt\\ &+\sum_{|\beta|\leq[ns]}\sum_{\alpha\in\mathbb{Z}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\bigtriangleup_h D^\beta(\sigma_\alpha v)(t,x)|^p}{|h|^{n+p\{ns\}}}dxdhdt\\ :=& I+II.\end{aligned} \tag{2.27}$$

We now estimate II. It is easy to see that

$$\begin{aligned}|\bigtriangleup_h D^\beta(\sigma_\alpha v)| &\lesssim \sum_{\beta_1+\beta_2=\beta}|\bigtriangleup_h(D^{\beta_1}\sigma_\alpha D^{\beta_2}v)|\\ &\leqslant \sum_{\beta_1+\beta_2=\beta}(|D^{\beta_1}\sigma_\alpha(\cdot+h)\ \bigtriangleup_h D^{\beta_2}v|+|(\bigtriangleup_h D^{\beta_1}\sigma_\alpha)D^{\beta_2}v|).\end{aligned} \tag{2.28}$$

Since $\operatorname{supp}\sigma_\alpha$ overlaps at most finitely many $\operatorname{supp}\sigma_\beta$ and $\sigma_\beta=\sigma_0(\cdot-\beta)$, $\beta\in\mathbb{Z}^n$, it follows from (2.28), $|D^{\beta_1}\sigma_\alpha|\lesssim 1$ and Hölder's inequality that

$$\begin{aligned}II \lesssim &\sum_{|\beta_1|,|\beta_2|\leq[ns]}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{\alpha\in\mathbb{Z}^n}|D^{\beta_1}\sigma_\alpha(x+h)|\frac{|\bigtriangleup_h D^{\beta_2}v(t,x)|^p}{|h|^{n+p\{ns\}}}dxdhdt\\ &+\sum_{|\beta|\leq[ns]}\sum_{\beta_1+\beta_2=\beta}\int_{\mathbb{R}^n}\frac{\|\bigtriangleup_h D^{\beta_1}\sigma_0\|^p_{L^\infty(\mathbb{R}^n)}}{|h|^{n+p\{ns\}}}dh\\ &\quad\times\sup_h\sum_{\alpha\in\mathbb{Z}^n}\int_{\mathbb{R}}\int_{B(0,\sqrt{n})\cup B(-h,\sqrt{n}))}|D^{\beta_2}v(t,x+\alpha)|^pdxdt\\ \lesssim &\sum_{|\beta|\leq[ns]}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\bigtriangleup_h D^\beta v(t,x)|^p}{|h|^{n+p\{ns\}}}dxdhdt\\ &+\|\sigma_\alpha\|^p_{B^{ns}_{\infty,p}}\sum_{|\beta|\leq[ns]}\int_{\mathbb{R}}\int_{\mathbb{R}^n}|D^\beta v(x)|^pdxdt\\ \lesssim &\|v\|^p_{L^p(\mathbb{R},B^{ns}_{p,p}(\mathbb{R}^n))}.\end{aligned} \tag{2.29}$$

Clearly, one has that

$$I\lesssim\|v\|^p_{L^p(\mathbb{R},B^{ns}_{p,p}(\mathbb{R}^n))}. \tag{2.30}$$

Collecting (2.27), (2.29) and (2.30), we have (2.23).

Case 2. $ns \in \mathbb{N}$. One can take an $s_1 < s$ such that $s_1 > 1/p$ and $ns_1 \notin \mathbb{N}$. Applying the conclusion as in Case 1, we get the result, as desired. $\square$

For the semi-group $S(t)$, we have the following Strichartz estimate (cf. [14]):

Proposition 2.4 *Let $n \geq 2$. $2 \leq p, \rho \leq 2n/(n-2)$ ($2 \leq p, \rho < \infty$ if $n = 2$), $2/\gamma(\cdot) = n(1/2 - 1/\cdot)$. We have*

$$\|S(t)u_0\|_{L^{\gamma(p)}(\mathbb{R}, L^p(\mathbb{R}^n))} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}, \tag{2.31}$$

$$\|\mathscr{A}F\|_{L^{\gamma(p)}(\mathbb{R}, L^p(\mathbb{R}^n))} \lesssim \|F\|_{L^{\gamma(\rho)'}(\mathbb{R}, L^{\rho'}(\mathbb{R}^n))}. \tag{2.32}$$

If p and ρ equal to $2n/(n-2)$, then (2.31) and (2.32) are said to be the endpoint Strichartz estimates. Using Proposition 2.4, we have

Proposition 2.5 *Let $p \geq 2 + 4/n := 2^*$. For any $s > n/2$, we have*

$$\left(\sum_{\alpha \in \mathbb{Z}^n} \| S(t)u_0\|^p_{L^\infty_{t,x}(\mathbb{R} \times Q_\alpha)} \right)^{1/p} \lesssim \|u_0\|_{H^s}. \tag{2.33}$$

Proof. For short, we write $\langle \partial_t \rangle = (I - \partial_t^2)^{1/2}$. By Lemma 2.3, for any $s_0 > 1/2^*$,

$$\begin{aligned} \left(\sum_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|^p_{L^\infty_{t,x}(\mathbb{R} \times Q_\alpha)} \right)^{1/p} &\lesssim \left(\sum_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|^{2^*}_{L^\infty_{t,x}(\mathbb{R} \times Q_\alpha)} \right)^{1/2^*} \\ &\lesssim \|\langle \partial_t \rangle^{s_0} S(t)u_0\|_{L^{2^*}(\mathbb{R}, B^{ns_0}_{2^*,2^*}(\mathbb{R}^n))}. \end{aligned} \tag{2.34}$$

We have

$$\|\langle \partial_t \rangle^{s_0} S(t)u_0\|^{2^*}_{L^{2^*}(\mathbb{R}, B^{ns_0}_{2^*,2^*}(\mathbb{R}^n))} = \sum_{k=0}^{\infty} 2^{ns_0 k 2^*} \|\langle \partial_t \rangle^{s_0} \triangle_k S(t)u_0\|^{2^*}_{L^{2^*}_{t,x}(\mathbb{R}^{1+n})}. \tag{2.35}$$

Using the dyadic decomposition to the time-frequency, we obtain that

$$\|\langle \partial_t \rangle^{s_0} \triangle_k S(t)u_0\|_{L^{2^*}_{t,x}} \lesssim \sum_{j=0}^{\infty} \|\mathscr{F}^{-1}_{t,x} \langle \tau \rangle^{s_0} \varphi_j(\tau) \mathscr{F}_t e^{\mathrm{i}t|\xi|^2_\pm} \varphi_k(\xi) \mathscr{F}_x u_0\|_{L^{2^*}_{t,x}}. \tag{2.36}$$

Noticing the fact that

$$(\mathscr{F}^{-1}_t \langle \tau \rangle^{s_0} \varphi_j(\tau)) \star e^{\mathrm{i}t|\xi|^2_\pm} = c\, e^{\mathrm{i}t|\xi|^2_\pm} \varphi_j(|\xi|^2_\pm) \langle |\xi|^2_\pm \rangle^{s_0}, \tag{2.37}$$

and using the Strichartz inequality and Plancherel's identity, one has that

$$\begin{aligned}\|\langle\partial_t\rangle^{s_0}\triangle_k S(t)u_0\|_{L^{2^*}_{t,x}} &\lesssim \sum_{j=0}^{\infty}\|S(t)\mathscr{F}_x^{-1}\langle|\xi|^2_\pm\rangle^{s_0}\varphi_j(|\xi|^2_\pm)\varphi_k(\xi)\mathscr{F}_x u_0\|_{L^{2^*}_{t,x}}\\ &\lesssim \sum_{j=0}^{\infty}\|\mathscr{F}_x^{-1}\langle|\xi|^2_\pm\rangle^{s_0}\varphi_j(|\xi|^2_\pm)\varphi_k(\xi)\mathscr{F}_x u_0\|_{L^2_x(\mathbb{R}^n)}\\ &\lesssim 2^{2s_0k}\sum_{j=0}^{\infty}\|\mathscr{F}_x^{-1}\varphi_j(|\xi|^2_\pm)\varphi_k(\xi)\mathscr{F}_x u_0\|_{L^2_x(\mathbb{R}^n)}. \qquad (2.38)\end{aligned}$$

Combining (2.35) and (2.38), together with Minkowski's inequality, we have

$$\begin{aligned}&\|\langle\partial_t\rangle^{s_0}S(t)u_0\|_{L^{2^*}(\mathbb{R},B^{ns_0}_{2^*,2^*}(\mathbb{R}^n))}\\ &\lesssim \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}2^{(n+2)s_0k2^*}\|\mathscr{F}^{-1}\varphi_j(|\xi|^2_\pm)\varphi_k\mathscr{F}u_0\|^{2^*}_{L^2_x(\mathbb{R}^n)}\right)^{1/2^*}\\ &\lesssim \sum_{j=0}^{\infty}\|\mathscr{F}^{-1}\varphi_j(|\xi|^2_\pm)\mathscr{F}u_0\|_{B^{(n+2)s_0}_{2,2^*}}. \qquad (2.39)\end{aligned}$$

In view of $H^{(n+2)s_0}\subset B^{(n+2)s_0}_{2,2^*}$ and Hölder's inequality, we have for any $\varepsilon>0$,

$$\begin{aligned}\sum_{j=0}^{\infty}\|\mathscr{F}^{-1}\varphi_j(|\xi|^2_\pm)\mathscr{F}u_0\|_{B^{(n+2)s_0}_{2,2^*}} &\lesssim \sum_{j=0}^{\infty}\|\mathscr{F}^{-1}\varphi_j(|\xi|^2_\pm)\mathscr{F}u_0\|_{H^{(n+2)s_0}}\\ &\lesssim \left(\sum_{j=0}^{\infty}2^{2j\varepsilon}\|\mathscr{F}^{-1}\varphi_j(|\xi|^2_\pm)\mathscr{F}u_0\|^2_{H^{(n+2)s_0}}\right)^{1/2}. \qquad (2.40)\end{aligned}$$

By Plancherel's identity, and $\mathrm{supp}\varphi_j(|\xi|^2_\pm)\subset\{\xi:\ ||\xi|^2_\pm|\in[2^{j-1},2^{j+1}]\}$, we easily see that

$$\begin{aligned}\left(\sum_{j=0}^{\infty}2^{2j\varepsilon}\|\mathscr{F}^{-1}\varphi_j(|\xi|^2_\pm)\mathscr{F}u_0\|^2_{H^{(n+2)s_0}}\right)^{1/2} &\lesssim \left(\sum_{j=0}^{\infty}\|\langle|\xi|^2_\pm\rangle^{\varepsilon}\varphi_j(|\xi|^2_\pm)\mathscr{F}u_0\|^2_{H^{(n+2)s_0}}\right)^{1/2}\\ &\lesssim \left(\sum_{j=0}^{\infty}\|\varphi_j(|\xi|^2_\pm)\mathscr{F}u_0\|^2_{H^{(n+2)s_0+2\varepsilon}}\right)^{1/2}\\ &\lesssim \|u_0\|_{H^{(n+2)s_0+2\varepsilon}}. \qquad (2.41)\end{aligned}$$

Taking s_0 such that $(n+2)s_0+2\varepsilon<s$, from (2.39)–(2.41) we have the result, as desired. □

Next, we consider the estimates for the maximal function based on the frequency-uniform decomposition method. This issue has some relations with the Strichartz estimates in modulation spaces. Recently, the Strichartz estimates have been generalized to various function spaces, for instance, in the

Wiener amalgam spaces [10,11]. Recall that in [32], we obtained the following Strichartz estimate for a class of dispersive semi-groups in modulation spaces:

$$U(t) = \mathscr{F}^{-1} e^{\mathrm{i}tP(\xi)} \mathscr{F}, \tag{2.42}$$

$P(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a real valued function, which satisfies the following decay estimate

$$\|U(t)f\|_{M^{\alpha}_{p,q}} \lesssim (1+|t|)^{-\delta} \|f\|_{M_{p',q}}, \tag{2.43}$$

where $2 \le p < \infty$, $\alpha = \alpha(p) \in \mathbb{R}$, $\delta = \delta(p) > 0$, α, δ are independent of $t \in \mathbb{R}$.

Proposition 2.6 *Let $U(t)$ satisfy* (2.43) *and* (2.44)*. We have for any $\gamma \ge 2 \vee (2/\delta)$,*

$$\|U(t)f\|_{L^{\gamma}(\mathbb{R}, M^{\alpha/2}_{p,1})} \lesssim \|f\|_{M_{2,1}}. \tag{2.44}$$

Recall that the hyperbolic Schrödinger semi-group $S(t) = e^{\mathrm{i}t\Delta_{\pm}}$ has the same decay estimate as that of the elliptic Schrödinger semi-group $e^{\mathrm{i}t\Delta}$:

$$\|S(t)u_0\|_{L^{\infty}(\mathbb{R}^n)} \lesssim |t|^{-n/2} \|u_0\|_{L^1(\mathbb{R}^n))}.$$

It follows that

$$\|S(t)u_0\|_{M_{\infty,1}} \lesssim |t|^{-n/2} \|u_0\|_{M_{1,1}}. \tag{2.45}$$

On the other hand, by Hausdorff-Young's and Hölder's inequalities we easily calculate that

$$\begin{aligned} \|\Box_k S(t)u_0\|_{L^{\infty}(\mathbb{R}^n)} &\lesssim \sum_{\ell \in \Lambda} \|\mathscr{F}^{-1} \sigma_{k+\ell} \mathscr{F} \Box_k S(t) u_0\|_{L^{\infty}(\mathbb{R}^n)} \\ &\lesssim \sum_{\ell \in \Lambda} \|\sigma_{k+\ell} \mathscr{F} \Box_k u_0\|_{L^1(\mathbb{R}^n)} \lesssim \|\Box_k u_0\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

where Λ is as in (2.9). It follows that

$$\|S(t)u_0\|_{M_{\infty,1}} \lesssim \|u_0\|_{M_{1,1}}. \tag{2.46}$$

Hence, in view of (2.45) and (2.46), we have

$$\|S(t)u_0\|_{M_{\infty,1}} \lesssim (1+|t|)^{-n/2} \|u_0\|_{M_{1,1}}. \tag{2.47}$$

By Plancherel's identity, one has that

$$\|S(t)u_0\|_{M_{2,1}} = \|u_0\|_{M_{2,1}}. \tag{2.48}$$

Hence, an interpolation between (2.47) and (2.48) yields (cf. [33]),

$$\|S(t)u_0\|_{M_{p,1}} \lesssim (1+|t|)^{-n(1/2-1/p)} \|u_0\|_{M_{p',1}}.$$

Applying Proposition 2.6, we immediately obtain that

Proposition 2.7 *Let $2 \le p < \infty$, $2/\gamma(p) = n(1/2 - 1/p)$. We have for any $\gamma \ge 2 \vee \gamma(p)$,*

$$\|S(t)u_0\|_{L^\gamma(\mathbb{R}, M^{\alpha/2}_{p,1})} \lesssim \|u_0\|_{M_{2,1}}. \tag{2.49}$$

In particular, if $p \ge 2 + 4/n := 2^*$, then

$$\|S(t)u_0\|_{L^p(\mathbb{R}, M^{\alpha/2}_{p,1})} \lesssim \|u_0\|_{M_{2,1}}. \tag{2.50}$$

Let $\Lambda = \{\ell \in \mathbb{Z}^n : \ \operatorname{supp} \sigma_\ell \cap \operatorname{supp} \sigma_0 \neq \varnothing\}$ be as in (2.9). Using the fact that $\Box_k \Box_{k+\ell} = 0$ if $\ell \notin \Lambda$, it is easy to see that (2.50) implies the following frequency-uniform estimates:

$$\| \Box_k S(t)u_0\|_{L^p_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \| \Box_k u_0\|_2, \quad k \in \mathbb{Z}^n. \tag{2.51}$$

Applying this estimate, we can get the following

Proposition 2.8 *Let $p \ge 2 + 4/n := 2^*$ For any $s > (n+2)/p$, we have*

$$\sum_{k\in\mathbb{Z}^n} \left(\sum_{\alpha\in\mathbb{Z}^n} \| \Box_k S(t)u_0\|^p_{L^\infty_{t,x}(\mathbb{R}\times Q_\alpha)} \right)^{1/p} \lesssim \|u_0\|_{M^s_{2,1}}. \tag{2.52}$$

Proof. Let us follow the proof of Proposition 2.5. Denote $\langle \partial_t \rangle = (I - \partial_t^2)^{1/2}$. By Lemma 2.3, for any $s_0 > 1/p$,

$$\begin{aligned} \sum_{k\in\mathbb{Z}^n} \left(\sum_{\alpha\in\mathbb{Z}^n} \| \Box_k S(t)u_0\|^p_{L^\infty_{t,x}(\mathbb{R}\times Q_\alpha)} \right)^{1/p} & \lesssim \sum_{k\in\mathbb{Z}^n} \|\langle \partial_t \rangle^{s_0} S(t) \Box_k u_0\|_{L^p(\mathbb{R}, B^{ns_0}_{p,p}(\mathbb{R}^n))} \\ & \lesssim \sum_{k\in\mathbb{Z}^n} \|\langle \partial_t \rangle^{s_0} S(t) \Box_k u_0\|_{L^p(\mathbb{R}, H^{ns_0}_{p}(\mathbb{R}^n))}, \end{aligned} \tag{2.53}$$

where we have used the fact that $H^{ns_0}_p(\mathbb{R}^n) \subset B^{ns_0}_{p,p}(\mathbb{R}^n)$. Since $\operatorname{supp}\sigma_k \subset B(k, \sqrt{n/2})$, applying Bernstein's multiplier estimate, we get that

$$\sum_{k\in\mathbb{Z}^n} \|\langle \partial_t \rangle^{s_0} S(t) \Box_k u_0\|_{L^p(\mathbb{R}, H^{ns_0}_{p}(\mathbb{R}^n))} \lesssim \sum_{k\in\mathbb{Z}^n} \langle k \rangle^{ns_0} \|\langle \partial_t \rangle^{s_0} S(t) \Box_k u_0\|_{L^p_{t,x}(\mathbb{R}^{1+n})}. \tag{2.54}$$

Similarly as in (2.38), using (2.51), we have

$$\|\langle \partial_t \rangle^{s_0} S(t) \Box_k u_0\|_{L^p_{t,x}(\mathbb{R}^{1+n})} \lesssim \sum_{j=0}^{\infty} \|\mathscr{F}_x^{-1} \langle |\xi|^2_\pm \rangle^{s_0} \varphi_j(|\xi|^2_\pm) e^{\mathrm{i}t|\xi|^2_\pm} \sigma_k(\xi) \mathscr{F}_x u_0\|_{L^p_{t,x}(\mathbb{R}^{1+n})}$$

$$\lesssim \sum_{j=0}^{\infty} \langle k\rangle^{2s_0} \|\mathscr{F}_x^{-1}\varphi_j(|\xi|^2_{\pm})\sigma_k(\xi)\mathscr{F}_x u_0\|_{L^2_x(\mathbb{R}^n)}. \quad (2.55)$$

In an analogous way as in (2.40) and (2.41), we obtain that

$$\sum_{j=0}^{\infty} \langle k\rangle^{2s_0} \|\mathscr{F}_x^{-1}\varphi_j(|\xi|^2_{\pm})\sigma_k(\xi)\mathscr{F}_x u_0\|_{L^2_x(\mathbb{R}^n)} \lesssim \langle k\rangle^{2s_0+2\varepsilon} \|\,\Box_k u_0\|_{L^2_x(\mathbb{R}^n)}. \quad (2.56)$$

Collecting (2.53)–(2.56), we have

$$\sum_{k\in\mathbb{Z}^n} \left(\sum_{\alpha\in\mathbb{Z}^n} \|\,\Box_k S(t)u_0\|^p_{L^\infty_{t,x}(\mathbb{R}\times Q_\alpha)}\right)^{1/p} \lesssim \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{(n+2)s_0+2\varepsilon} \|\,\Box_k u_0\|_{L^2_x(\mathbb{R}^n)}. \quad (2.57)$$

Hence, by (2.57) we have (2.52). $\Box$

Using the ideas as in Lemma 2.3 and Proposition 2.5, we can show the following

Proposition 2.9 *Let $p \geq 2+4/n := 2^*$. Let $2^* \leq r, q \leq \infty$, $s_0 > 1/2^* - 1/q$, $s_1 > n(1/2^* - 1/r)$. Then we have*

$$\left(\sum_{\alpha\in\mathbb{Z}^n} \|\,S(t)u_0\|^p_{L^q(\mathbb{R},\, L^r(Q_\alpha))}\right)^{1/p} \lesssim \|u_0\|_{H^{s_1+2s_0}}. \quad (2.58)$$

In particular, for any $q, p \geq 2^*$, $s > n/2 - 2/q$,

$$\left(\sum_{\alpha\in\mathbb{Z}^n} \|\,S(t)u_0\|^p_{L^q(\mathbb{R},\, L^\infty(Q_\alpha))}\right)^{1/p} \lesssim \|u_0\|_{H^s}. \quad (2.59)$$

Sketch of Proof. In view of $\ell^{2^*} \subset \ell^p$, it suffices to consider the case $p = 2^*$. Using the inclusions $H^{s_0}_p(\mathbb{R}) \subset L^q(\mathbb{R})$ and $B^{s_1}_{p,p}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$, we have

$$\|u\|_{L^q(\mathbb{R},\, L^r(Q_\alpha))} \lesssim \|(I-\partial_t^2)^{s_0/2}\sigma_\alpha u\|_{L^p(\mathbb{R}, B^{s_1}_{p,p}(\mathbb{R}^n))}. \quad (2.60)$$

Using the same way as in Lemma 2.3, we can show that

$$\left(\sum_{\alpha\in\mathbb{Z}^n} \|u\|^p_{L^q(\mathbb{R},\, L^r(Q_\alpha))}\right)^{1/p} \lesssim \|(I-\partial_t^2)^{s_0/2} u\|_{L^p(\mathbb{R}, B^{s_1}_{p,p}(\mathbb{R}^n))}. \quad (2.61)$$

One can repeat the procedures as in the proof of Lemma 2.3 to conclude that

$$\sum_{\alpha\in\mathbb{Z}^n} \|(I-\partial_t^2)^{s_0/2}\sigma_\alpha S(t)u_0\|^p_{L^p(\mathbb{R}, B^{s_1}_{p,p}(\mathbb{R}^n))} \lesssim \sum_{j=0}^{\infty} \|\mathscr{F}^{-1}\varphi_j(|\xi|^2_{\pm})\mathscr{F}u_0\|_{H^{s_1+2s_0}(\mathbb{R}^n)}. \quad (2.62)$$

Applying an analogous way as in the proof of Proposition 2.5,

$$\sum_{j=0}^{\infty} \|\mathscr{F}^{-1}\varphi_j(|\xi|_{\pm}^2)\mathscr{F}u_0\|_{H^{s_1+2s_0}(\mathbb{R}^n)} \lesssim \|u_0\|_{H^{s_1+2s_0+2\varepsilon}}. \tag{2.63}$$

Collecting (2.61) and (2.63), we immediately get (2.58). □

Proposition 2.10 *For any $q \geq p \geq 2^*$, $s > (n+2)/p - 2/q$,*

$$\sum_{k\in\mathbb{Z}^n} \left(\sum_{\alpha\in\mathbb{Z}^n} \| \square_k S(t)u_0\|^p_{L^q(\mathbb{R},\, L^\infty(Q_\alpha))} \right)^{1/p} \lesssim \|u_0\|_{M^s_{2,1}}. \tag{2.64}$$

3 Global-local estimates on time-space

3.1 Time-global and space-local Strichartz estimates

We need some modifications to the Strichartz estimates, which are global on time variable and local on spatial variable. We always denote by $S(t)$ and $\mathscr{A}$ the generalized Schrödinger semi-group and the integral operator as in (1.9).

Proposition 3.1 *Let $n \geq 3$. Then we have*

$$\sup_{\alpha\in\mathbb{Z}^n} \|S(t)u_0\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \|u_0\|_2, \tag{3.1}$$

$$\sup_{\alpha\in\mathbb{Z}^n} \|\mathscr{A}F\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \sum_{\alpha\in\mathbb{Z}^n} \|F\|_{L^1_t L^2_x(\mathbb{R}\times Q_\alpha)}. \tag{3.2}$$

$$\sup_{\alpha\in\mathbb{Z}^n} \|\mathscr{A}F\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \sum_{\alpha\in\mathbb{Z}^n} \|F\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)}. \tag{3.3}$$

Proof. In view of Hölder's inequality and the endpoint Strichartz estimate,

$$\begin{aligned} \|S(t)u_0\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} &\lesssim \|S(t)u_0\|_{L^2_t L^{2n/(n-2)}_x(\mathbb{R}\times Q_\alpha)} \\ &\leq \|S(t)u_0\|_{L^2_t L^{2n/(n-2)}_x(\mathbb{R}\times \mathbb{R}^n)} \\ &\lesssim \|u_0\|_{L^2_x(\mathbb{R}^n)}. \end{aligned} \tag{3.4}$$

Using the above ideas and the following Strichartz estimate

$$\|\mathscr{A}F\|_{L^2_t L^{2n/(n-2)}_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L^1_t L^2_x(\mathbb{R}\times\mathbb{R}^n)}, \tag{3.5}$$

$$\|\mathscr{A}F\|_{L^2_t L^{2n/(n-2)}_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|F\|_{L^2_t L^{2n/(n+2)}_x(\mathbb{R}\times\mathbb{R}^n)}, \tag{3.6}$$

one can easily get (3.2) and (3.3). □

Since the endpoint Strichartz estimates used in the proof of Proposition 3.1 only holds for $n \geq 3$, it is not clear for us if (3.1) still hold for $n = 2$. This is why we have an additional condition that $u_0 \in \dot{H}^{-1/2}$ is small in 2D. However, we have the following (see [17])

Proposition 3.2 *Let $n = 2$. Then we have for any $1 \leq r < 4/3$,*

$$\sup_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \min\left(\|(-\Delta)^{-1/4}u_0\|_2,\ \|u_0\|_{L^2\cap L^r(\mathbb{R}^n)}\right). \tag{3.7}$$

In the low frequency case, one easily sees that (3.7) is strictly weak than (3.1).

Proof. By Lemma 3.4, it suffices to show

$$\sup_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \|u_0\|_{L^2\cap L^r(\mathbb{R}^n)}. \tag{3.8}$$

Using the unitary property in L^2 and the $L^p - L^{p'}$ decay estimates of $S(t)$, we have

$$\|S(t)u_0\|_{L^2_x(Q_\alpha)} \lesssim (1+|t|)^{1-2/r}\|u_0\|_{L^2\cap L^r(\mathbb{R}^n)}. \tag{3.9}$$

Taking the L^2_t norm in both sides of (3.9), we immediately get (3.8). Hence, the result follows. □

Proposition 3.3 *Let $n = 2$. Then we have*

$$\sup_{\alpha \in \mathbb{Z}^n} \|\mathscr{A}F\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^2} \|F\|_{L^1_t L^2_x(\mathbb{R}\times Q_\alpha)}. \tag{3.10}$$

Proof. We notice that

$$\|S(t)f\|_{L^2_x(Q_\alpha)} \lesssim (1+|t|)^{-1}\|f\|_{L^1_x\cap L^2_x(\mathbb{R}^n)}. \tag{3.11}$$

It follows that

$$\|\mathscr{A}F\|_{L^2_x(Q_\alpha)} \lesssim \int_{\mathbb{R}} (1+|t-\tau|)^{-1}\|F(\tau)\|_{L^1_x\cap L^2_x(\mathbb{R}^n)}d\tau. \tag{3.12}$$

Using Young's inequality, one has that

$$\|\mathscr{A}F\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \|F\|_{L^1(\mathbb{R},\, L^1_x\cap L^2_x(\mathbb{R}^n))}. \tag{3.13}$$

In view of Hölder's inequality, (3.13) yields the result, as desired. □

Kenig, Ponce and Vega [16,17] obtained the local smooth effect estimates for the Schrödinger group $e^{\mathrm{i}t\Delta}$, and their results can also be developed to the non-elliptical Schrödinger group $e^{\mathrm{i}t\Delta_\pm}$ ([18]). On the basis of their results and Proposition 3.1, we can obtain a time-global version of the local smooth effect estimates with the nonhomogeneous derivative $(I-\Delta)^{1/2}$ instead of homogeneous derivative ∇, which is useful to control the low frequency parts of the nonlinearity.

Lemma 3.4 *([16]) Let Ω be an open set in $\mathbb{R}^n$, ϕ be a $C^1(\Omega)$ function such that $\nabla\phi(\xi)\neq 0$ for any $\xi\in\Omega$. Assume that there is $N\in\mathbb{N}$ such that for any $\bar{\xi}:=(\xi_1,...,\xi_{n-1})\in\mathbb{R}^{n-1}$ and $r\in\mathbb{R}$, the equation $\phi(\xi_1,...,\xi_k,x,\xi_{k+1},...,\xi_{n-1})=r$ has at most N solutions. For $a(x,s)\in L^\infty(\mathbb{R}^n\times\mathbb{R})$ and $f\in\mathscr{S}(\mathbb{R}^n)$, we denote*

$$W(t)f(x)=\int_\Omega e^{\mathrm{i}(t\phi(\xi)+x\xi)}a(x,\phi(\xi))\hat{f}(\xi)d\xi. \tag{3.14}$$

Then for $n\geq 2$, we have

$$\|W(t)f\|_{L^2_{t,x}(\mathbb{R}\times B(0,R))}\leq CNR^{1/2}\||\nabla\phi|^{-1/2}\hat{f}\|_{L^2(\Omega)}. \tag{3.15}$$

Corollary 3.5 *Let $n\geq 3$, $S(t)=e^{\mathrm{i}t\Delta_\pm}$. We have*

$$\sup_{\alpha\in\mathbb{Z}^n}\|S(t)u_0\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)}\lesssim\|u_0\|_{H^{-1/2}}, \tag{3.16}$$

$$\|\mathscr{A}f\|_{L^\infty(\mathbb{R},H^{1/2})}\lesssim\sum_{\alpha\in\mathbb{Z}^n}\|f\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)}. \tag{3.17}$$

For $n=2$, (3.17) also holds if one substitutes $H^{1/2}$ by $\dot{H}^{1/2}$.

Proof. Let $\Omega=\mathbb{R}^n\setminus B(0,1)$, $\phi(\xi)=|\xi|^2_\pm$ and ψ be as in (1.24), $a(x,s)=1-\psi(s)$ in Lemma 3.4. Taking $W(t):=S(t)\mathscr{F}^{-1}(1-\psi)\mathscr{F}$, from (3.15) we have

$$\sup_{\alpha\in\mathbb{Z}^n}\|S(t)\mathscr{F}^{-1}(1-\psi)\mathscr{F}u_0\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)}\lesssim\||\xi|^{-1/2}\hat{u}_0\|_{L^2_\xi(\mathbb{R}^n\setminus B(0,1))}. \tag{3.18}$$

It follows from Proposition 3.1 that

$$\begin{aligned}\|S(t)\mathscr{F}^{-1}\psi\mathscr{F}u_0\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)}&\lesssim\|\mathscr{F}^{-1}\psi\mathscr{F}u_0\|_{L^2_x(\mathbb{R}^n)}\\&\lesssim\|\hat{u}_0\|_{L^2_\xi(B(0,2))}.\end{aligned} \tag{3.19}$$

From (3.18) and (3.19) we have (3.16), as desired. (3.17) is the dual version of (3.16). □

When $n = 2$, it is known that for the elliptic case, the endpoint Strichartz estimate holds for the radial function (cf. [28]). So, Corollary 3.5 also holds for the radial function u_0 in the elliptic case. The following local smooth effect estimates for the nonhomogeneous part of the solutions of the Schrödinger equation is also due to Kenig, Ponce and Vega [17] [2].

Proposition 3.6 *Let $n \geq 2$, $S(t) = e^{\mathrm{i}t\Delta_\pm}$. We have*

$$\sup_{\alpha\in\mathbb{Z}^n} \|\nabla \mathscr{A} f\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \lesssim \sum_{\alpha\in\mathbb{Z}^n} \|f\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)}. \tag{3.20}$$

4 Proof of Theorem 1.1

Lemma 4.1 *(Sobolev Inequality). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^m$, $m, \ell \in \mathbb{N}\cup\{0\}$, $1 \leq r, p, q \leq \infty$. Assume that*

$$\frac{\ell}{m} \leq \theta \leq 1, \quad \frac{1}{p} - \frac{\ell}{n} = \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\theta}{q}.$$

Then we have

$$\sum_{|\beta|=\ell} \|D^\beta u\|_{L^p(\Omega)} \lesssim \|u\|^{1-\theta}_{L^q(\Omega)} \|u\|^{\theta}_{W^m_r(\Omega)}, \tag{4.1}$$

where $\|u\|_{W^m_r(\Omega)} = \sum_{|\beta|\leq m} \|u\|_{L^r(\Omega)}$.

Proof of Theorem 1.1. In order to illustrate our ideas in an exact way, we first consider a simple case $s = [n/2] + 5/2$ and there is no difficulty to generalize the proof to the case $s > n/2+3/2$, $s+1/2 \in \mathbb{N}$. We assume without loss of generality that

$$F(u,\, \bar{u},\, \nabla u,\, \nabla\bar{u}) := F(u, \nabla u) = \sum_{\Lambda_{\kappa,\nu}} c_{\kappa\nu_1\ldots\nu_n} u^\kappa u^{\nu_1}_{x_1} \ldots u^{\nu_n}_{x_n}, \tag{4.2}$$

where

$$\Lambda_{\kappa,\nu} = \{(\kappa, \nu_1, ..., \nu_n) :\ m + 1 \leq \kappa + \nu_1 + ... + \nu_n \leq M + 1\}.$$

Since we only use the Sobolev norm to control the nonlinear terms, $\bar{u}$ and u have the same norm, whence, the general cases can be handled in the same way. Denote

$$\lambda_1(v) := \|v\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))},$$

[2] In [17], the result was stated for the elliptic case, however, their result is also adapted to the non-elliptic cases.

$$\lambda_2(v) := \|v\|_{\ell^{2^*}_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))},$$
$$\lambda_3(v) := \|v\|_{\ell^{2^*}_\alpha(L^{2m}_t L^\infty_x(\mathbb{R}\times Q_\alpha))}.$$

Put

$$\mathscr{D}_n = \left\{ u : \sum_{|\beta|\le[n/2]+3} \lambda_1(D^\beta u) + \sum_{|\beta|\le 1}\sum_{i=2,3} \lambda_i(D^\beta u) \le \varrho \right\}. \tag{4.3}$$

We consider the mapping

$$\mathscr{T} : u(t) \to S(t)u_0 - \mathrm{i}\mathscr{A} F(u, \nabla u), \tag{4.4}$$

and we show that $\mathscr{T} : \mathscr{D}_n \to \mathscr{D}_n$ is a contraction mapping for any $n \ge 2$.

Step 1. For any $u \in \mathscr{D}_n$, we estimate $\lambda_1(D^\beta \mathscr{T} u)$, $|\beta| \le 3+[n/2]$. We consider the following three cases.

Case 1. $n \ge 3$ and $1 \le |\beta| \le 3 + [n/2]$. In view of Corollary 3.5 and Proposition 3.6, we have for any β, $1 \le |\beta| \le 3 + [n/2]$,

$$\begin{aligned}\lambda_1(D^\beta \mathscr{T} u) &\lesssim \|S(t)D^\beta u_0\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} + \sum_{\Lambda_{\kappa,\nu}} \|\mathscr{A} D^\beta(u^\kappa u^{\nu_1}_{x_1}...u^{\nu_n}_{x_n})\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \\ &\lesssim \|u_0\|_{H^s} + \sum_{|\beta|\le 2+[n/2]} \sum_{\Lambda_{\kappa,\nu}} \sum_{\alpha\in\mathbb{Z}^n} \|D^\beta(u^\kappa u^{\nu_1}_{x_1}...u^{\nu_n}_{x_n})\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)}.\end{aligned} \tag{4.5}$$

For simplicity, we can further assume that $u^\kappa u^{\nu_1}_{x_1}...u^{\nu_n}_{x_n} = u^\kappa u^\nu_{x_1}$ in (4.5) and the general case can be treated in an analogous way[3]. So, one can rewrite (4.5) as

$$\sum_{1\le|\beta|\le 3+[n/2]} \lambda_1(D^\beta \mathscr{T} u) \lesssim \|u_0\|_{H^s} + \sum_{|\beta|\le 2+[n/2]} \sum_{\Lambda_{\kappa,\nu}} \sum_{\alpha\in\mathbb{Z}^n} \|D^\beta(u^\kappa u^\nu_{x_1})\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)}. \tag{4.6}$$

It is easy to see that

$$|D^\beta(u^\kappa u^\nu_{x_1})| \lesssim \sum_{\beta_1+...+\beta_{\kappa+\nu}=\beta} |D^{\beta_1}u....D^{\beta_\kappa}u D^{\beta_{\kappa+1}}u_{x_1}...D^{\beta_{\kappa+\nu}}u_{x_1}|. \tag{4.7}$$

By Hölder's inequality,

$$\|D^\beta(u^\kappa u^\nu_{x_1})\|_{L^2_x(Q_\alpha)} \lesssim \sum_{\beta_1+...+\beta_{\kappa+\nu}=\beta} \prod_{i=1}^{\kappa} \|D^{\beta_i}u\|_{L^{p_i}_x(Q_\alpha)} \prod_{i=\kappa+1}^{\kappa+\nu} \|D^{\beta_i}u_{x_1}\|_{L^{p_i}_x(Q_\alpha)}, \tag{4.8}$$

---

[3] One can see below for a general treating.

where

$$p_i = \begin{cases} 2|\beta|/|\beta_i|, & |\beta_i| \geq 1, \\ \infty, & |\beta_i| = 0. \end{cases}$$

It is easy to see that for $\theta_i = |\beta_i|/|\beta|$,

$$\frac{1}{p_i} - \frac{|\beta_i|}{n} = \theta_i \left(\frac{1}{2} - \frac{|\beta|}{n} \right) + \frac{1-\theta_i}{\infty}.$$

Using Sobolev's inequality, one has that for $B_\alpha := \{x : \ |x-\alpha| \leq \sqrt{n}\}$,

$$\|D^{\beta_i} u\|_{L^{p_i}_x(Q_\alpha)} \leq \|D^{\beta_i} u\|_{L^{p_i}_x(B_\alpha)} \lesssim \|u\|^{1-\theta_i}_{L^\infty_x(B_\alpha)} \|u\|^{\theta_i}_{W^{|\beta|}_2(B_\alpha)}, \quad i = 1, ..., \kappa; \tag{4.9}$$

$$\|D^{\beta_i} u_{x_1}\|_{L^{p_i}_x(Q_\alpha)} \lesssim \|u_{x_1}\|^{1-\theta_i}_{L^\infty_x(B_\alpha)} \|u_{x_1}\|^{\theta_i}_{W^{|\beta|}_2(B_\alpha)}, \quad i = \kappa+1, ..., \kappa+\nu. \tag{4.10}$$

Since

$$\sum_{i=1}^{\kappa+\nu} \theta_i = 1, \quad \sum_{i=1}^{\kappa+\nu} (1-\theta_i) = \kappa + \nu - 1,$$

by (4.8)–(4.10) we have

$$\begin{aligned} \|D^\beta(u^\kappa u^\nu_{x_1})\|_{L^2_x(Q_\alpha)} \lesssim & \sum_{|\beta| \leq 2+[n/2]} (\|u\|_{W^{|\beta|}_2(B_\alpha)} + \|u_{x_1}\|_{W^{|\beta|}_2(B_\alpha)}) \\ & \times (\|u\|^{\kappa+\nu-1}_{L^\infty_x(B_\alpha)} + \|u_{x_1}\|^{\kappa+\nu-1}_{L^\infty_x(B_\alpha)}) \\ \lesssim & \sum_{|\gamma| \leq 3+[n/2]} \|D^\gamma u\|_{L^2_x(B_\alpha)} \sum_{|\beta| \leq 1} \|D^\beta u\|^{\kappa+\nu-1}_{L^\infty_x(B_\alpha)}. \end{aligned} \tag{4.11}$$

It follows from (4.11) and $\ell^{2^*} \subset \ell^{\kappa+\nu-1}$ that

$$\begin{aligned} & \sum_{|\beta| \leq 2+[n/2]} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta(u^\kappa u^\nu_{x_1})\|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \\ & \lesssim \sum_{\alpha \in \mathbb{Z}^n} \sum_{|\gamma| \leq 3+[n/2]} \|D^\gamma u\|_{L^2_{t,x}(\mathbb{R} \times B_\alpha)} \sum_{|\beta| \leq 1} \|D^\beta u\|^{\kappa+\nu-1}_{L^\infty_{t,x}(\mathbb{R} \times B_\alpha)} \\ & \lesssim \sum_{|\gamma| \leq 3+[n/2]} \lambda_1(D^\gamma u) \sum_{|\beta| \leq 1} \lambda_2(D^\beta u)^{\kappa+\nu-1} \lesssim \varrho^{\kappa+\nu}. \end{aligned} \tag{4.12}$$

Hence, in view of (4.6) and (4.12) we have

$$\sum_{1 \leq |\beta| \leq 3+[n/2]} \lambda_1(D^\beta \mathscr{T} u) \lesssim \|u_0\|_{H^s} + \sum_{\kappa+\nu=m+1}^{M+1} \varrho^{\kappa+\nu}. \tag{4.13}$$

Case 2. $n \geq 3$ and $|\beta| = 0$. By Corollary 3.5, the local Strichartz estimate (3.2) and Hölder's inequality,

$$\begin{aligned} \lambda_1(\mathscr{T} u) & \lesssim \|S(t)u_0\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))} + \|\mathscr{A} F(u, \nabla u)\|_{\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))} \\ & \lesssim \|u_0\|_2 + \sum_{\alpha \in \mathbb{Z}^n} \|F(u, \nabla u)\|_{L^1_t L^2_x(\mathbb{R} \times Q_\alpha)} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \|u_0\|_2 + \sum_{\Lambda_{\kappa,\nu}} \sum_{\alpha\in\mathbb{Z}^n} \|u^\kappa u_{x_1}^{\nu_1}...u_{x_n}^{\nu_n}\|_{L^1_t L^2_x(\mathbb{R}\times Q_\alpha)} \\
&\lesssim \|u_0\|_2 + \sum_{\kappa+\nu=m+1}^{M+1} \sum_{|\gamma|\le 1} \sup_{\alpha\in\mathbb{Z}^n} \|D^\gamma u\|_{L^2_{t,x}(\mathbb{R}\times Q_\alpha)} \\
&\qquad\qquad \times \sum_{|\beta|\le 1} \sum_{\alpha\in\mathbb{Z}^n} \|D^\beta u\|^{\kappa+\nu-1}_{L_t^{2(\kappa+\nu-1)} L^\infty_x(\mathbb{R}\times Q_\alpha)} \\
&\lesssim \|u_0\|_2 + \sum_{\kappa+\nu=m+1}^{M+1} \sum_{|\gamma|\le 1} \lambda_1(D^\gamma u) \sum_{i=2,3} \sum_{|\beta|\le 1} \lambda_i(D^\beta u)^{\kappa+\nu-1} \\
&\lesssim \|u_0\|_2 + \sum_{\kappa+\nu=m+1}^{M+1} \varrho^{\kappa+\nu}.
\end{aligned} \tag{4.14}
$$

Case 3. $n=2$, $|\beta|=0$. By Propositions 3.2 and 3.3, we have

$$
\lambda_1(\mathscr{T}u) \lesssim \|u_0\|_{\dot H^{-1/2}} + \sum_{\alpha\in\mathbb{Z}^n} \|F(u,\nabla u)\|_{L^1_t L^2_x(\mathbb{R}\times Q_\alpha)}. \tag{4.15}
$$

Using the same way as in Case 2, we have

$$
\lambda_1(\mathscr{T}u) \lesssim \|u_0\|_{\dot H^{-1/2}} + \sum_{\kappa+\nu=m+1}^{M+1} \varrho^{\kappa+\nu}. \tag{4.16}
$$

Step 2. We consider the estimates of $\lambda_2(D^\beta \mathscr{T}u)$, $|\beta|\le 1$. Using the estimates of the maximal function as in Proposition 2.5, we have for $|\beta|\le 1$, $0<\varepsilon\ll 1$,

$$
\begin{aligned}
\lambda_2(D^\beta \mathscr{T}u) &\lesssim \|S(t)D^\beta u_0\|_{\ell^{2*}_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))} + \|\mathscr{A} D^\beta F(u,\nabla u)\|_{\ell^{2*}_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))} \\
&\lesssim \|D^\beta u_0\|_{H^{n/2+\varepsilon}} + \sum_{|\beta|\le 1} \|D^\beta F(u,\nabla u)\|_{L^1(\mathbb{R},H^{n/2+\varepsilon}(\mathbb{R}^n))} \\
&\lesssim \|u_0\|_{H^{n/2+1+\varepsilon}} + \sum_{|\beta|\le[n/2]+2} \sum_{\alpha\in\mathbb{Z}^n} \|D^\beta F(u,\nabla u)\|_{L^1_t L^2_x(\mathbb{R}\times Q_\alpha)}.
\end{aligned} \tag{4.17}
$$

Applying the same way as in Step 1, for any $|\beta|\le[n/2]+2$,

$$
\|D^\beta F(u,\nabla u)\|_{L^2_x(Q_\alpha)} \lesssim \sum_{\kappa+\nu=m+1}^{M+1} \sum_{|\beta|\le 1} \|D^\beta u\|^{\kappa+\nu-1}_{L^\infty_x(B_\alpha)} \sum_{|\gamma|\le 3+[n/2]} \|D^\gamma u\|_{L^2_x(B_\alpha)}. \tag{4.18}
$$

By Hölder's inequality, we have from (4.18) that

$$
\begin{aligned}
\|D^\beta F(u,\nabla u)\|_{L^1_t L^2_x(\mathbb{R}\times Q_\alpha)} \lesssim &\sum_{\kappa+\nu=m+1}^{M+1} \sum_{|\gamma|\le 3+[n/2]} \|D^\gamma u\|_{L^2_{t,x}(\mathbb{R}\times B_\alpha)} \\
&\times \sum_{|\beta|\le 1} \|D^\beta u\|^{\kappa+\nu-1}_{L_t^{2(\kappa+\nu-1)} L^\infty_x(\mathbb{R}\times B_\alpha)}.
\end{aligned} \tag{4.19}
$$

Summarizing (4.19) over all $\alpha \in \mathbb{Z}^n$, we have for any $|\beta| \le 2 + [n/2]$,

$$
\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}^n} \|D^\beta F(u,\nabla u)\|_{L^1_t L^2_x(\mathbb{R}\times Q_\alpha)} \\
&\lesssim \sum_{\kappa+\nu=m+1}^{M+1} \sum_{|\gamma|\le 3+[n/2]} \lambda_1(D^\gamma u) \sum_{|\beta|\le 1} \sum_{\alpha\in\mathbb{Z}^n} \|D^\beta u\|^{\kappa+\nu-1}_{L^{2(\kappa+\nu-1)}_t L^\infty_x(\mathbb{R}\times B_\alpha)} \\
&\lesssim \sum_{\kappa+\nu=m+1}^{M+1} \sum_{|\gamma|\le 3+[n/2]} \lambda_1(D^\gamma u) \sum_{|\beta|\le 1} \sum_{\alpha\in\mathbb{Z}^n} \|D^\beta u\|^{\kappa+\nu-1}_{(L^{2m}_t L^\infty_x)\cap L^\infty_{t,x}(\mathbb{R}\times B_\alpha)} \\
&\lesssim \sum_{\kappa+\nu=m+1}^{M+1} \sum_{|\gamma|\le 3+[n/2]} \lambda_1(D^\gamma u) \sum_{|\beta|\le 1} (\lambda_2(D^\beta u)^{\kappa+\nu-1} + \lambda_3(D^\beta u)^{\kappa+\nu-1}) \\
&\lesssim \sum_{\kappa+\nu=m+1}^{M+1} \varrho^{\kappa+\nu}.
\end{aligned} \tag{4.20}
$$

Combining (4.17) with (4.20), we obtain that

$$
\sum_{|\beta|\le 1} \lambda_2(D^\beta \mathscr{T}u) \lesssim \|u_0\|_{H^{n/2+1+\varepsilon}} + \sum_{\kappa+\nu=m+1}^{M+1} \varrho^{\kappa+\nu}. \tag{4.21}
$$

Step 3. We estimate $\lambda_3(D^\beta \mathscr{T}u)$, $|\beta| \le 1$. In view of Proposition 2.9, one has that

$$
\begin{aligned}
\lambda_3(D^\beta \mathscr{T}u) &\lesssim \|S(t)D^\beta u_0\|_{\ell^{2*}_\alpha(L^{2m}_t L^\infty_x(\mathbb{R}\times Q_\alpha))} + \|\mathscr{A} D^\beta F(u,\nabla u)\|_{\ell^{2*}_\alpha(L^{2m}_t L^\infty_x(\mathbb{R}\times Q_\alpha))} \\
&\lesssim \|D^\beta u_0\|_{H^{n/2-1/m+\varepsilon}} + \sum_{|\beta|\le 1} \|D^\beta F(u,\nabla u)\|_{L^1(\mathbb{R}, H^{n/2-1/m+\varepsilon}(\mathbb{R}^n))} \\
&\lesssim \|u_0\|_{H^{n/2+1}} + \sum_{|\beta|\le [n/2+2]} \sum_{\alpha\in\mathbb{Z}^n} \|D^\beta F(u,\nabla u)\|_{L^1_t L^2_x(\mathbb{R}\times Q_\alpha)},
\end{aligned} \tag{4.22}
$$

which reduces to the case as in (4.17).

Therefore, collecting the estimates as in Steps 1–3, we have for $n \ge 3$,

$$
\sum_{|\beta|\le 3+[n/2]} \lambda_1(D^\beta \mathscr{T}u) + \sum_{i=2,3} \sum_{|\beta|\le 1} \lambda_i(D^\beta \mathscr{T}u) \lesssim \|u_0\|_{H^s} + \sum_{\kappa+\nu=m+1}^{M+1} \varrho^{\kappa+\nu}, \tag{4.23}
$$

and for $n \ge 2$,

$$
\sum_{|\beta|\le 4} \lambda_1(D^\beta \mathscr{T}u) + \sum_{i=2,3} \sum_{|\beta|\le 1} \lambda_i(D^\beta \mathscr{T}u) \lesssim \|u_0\|_{H^{7/2}\cap \dot{H}^{-1/2}} + \sum_{\kappa+\nu=m+1}^{M+1} \varrho^{\kappa+\nu}. \tag{4.24}
$$

It follows that for $n \ge 3$, $\mathscr{T} : \mathscr{D}_n \to \mathscr{D}_n$ is a contraction mapping if ϱ and $\|u_0\|_{H^s}$ are small enough (similarly for $n = 2$).

Before considering the case $s > n/2 + 3/2$, we first establish a nonlinear mapping estimate:

Lemma 4.2 *Let $n \geq 2$, $s > 0$, $K \in \mathbb{N}$. Let $1 \leq p, p_i, q, q_i \leq \infty$ satisfy $1/p = 1/p_1 + (K-1)/p_2$ and $1/q = 1/q_1 + (K-1)/q_2$. We have*

$$\|v_1 ... v_K\|_{\ell^{1,s}_\triangle \ell^1_\alpha (L^q_t L^p_x(\mathbb{R}\times Q_\alpha))} \lesssim \sum_{k=1}^K \|v_k\|_{\ell^{1,s}_\triangle \ell^\infty_\alpha (L^{q_1}_t L^{p_1}_x(\mathbb{R}\times Q_\alpha))} \times \prod_{i\neq k,\, i=1,...,K} \|v_i\|_{\ell^1_\triangle \ell^{K-1}_\alpha (L^{q_2}_t L^{p_2}_x(\mathbb{R}\times Q_\alpha))}. \tag{4.25}$$

Proof. Denote $S_r u = \sum_{j\leq r} \triangle u$. We have

$$v_1 ... v_K = \sum_{r=-1}^{\infty} (S_{r+1}v_1 ... S_{r+1}v_K - S_r v_1 ... S_r v_K), \tag{4.26}$$

where we assume that $S_{-1}v \equiv 0$. Recall the identity,

$$\prod_{k=1}^K a_k - \prod_{k=1}^K b_k = \sum_{k=1}^K (a_k - b_k) \prod_{i\leq k-1} b_i \prod_{i\geq k+1} a_i, \tag{4.27}$$

where we assume that $\prod_{i\leq 0} a_i = \prod_{i\geq K+1} \equiv 1$. We have

$$v_1 ... v_K = \sum_{r=-1}^{\infty} \sum_{k=1}^K \triangle_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^K S_{r+1} v_i. \tag{4.28}$$

Hence, it follows that

$$\begin{aligned} &\|v_1 ... v_K\|_{\ell^{1,s}_\triangle \ell^1_\alpha (L^q_t L^p_x(\mathbb{R}\times Q_\alpha))} \\ &= \sum_{j=0}^{\infty} 2^{sj} \sum_{\alpha\in\mathbb{Z}^n} \|\triangle_j (v_1 ... v_K)\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)} \\ &\lesssim \sum_{k=1}^K \sum_{j=0}^{\infty} 2^{sj} \sum_{\alpha\in\mathbb{Z}^n} \sum_{r=-1}^{\infty} \left\| \triangle_j (\triangle_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^K S_{r+1} v_i) \right\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)}. \end{aligned} \tag{4.29}$$

Using the support property of $\widehat{\triangle_r v}$ and $\widehat{S_r v}$, we see that

$$\triangle_j (\triangle_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^K S_{r+1} v_i) \equiv 0, \quad j > r + C. \tag{4.30}$$

Using the fact $\| \int f d\mu\|_X \leq \int \|f\|_X d\mu$, one has that

$$\sum_{\alpha\in\mathbb{Z}^n} \|\triangle_j f\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)} \leq \sum_{\alpha\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |\mathscr{F}^{-1}\varphi_j(y)| \|f(t, x-y)\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)} dy$$

$$\leq \sup_{y\in\mathbb{R}^n} \sum_{\alpha\in\mathbb{Z}^n} \|f(t,x-y)\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)} \int_{\mathbb{R}^n} |\mathscr{F}^{-1}\varphi_j(y)|dy$$
$$\lesssim \sum_{\alpha\in\mathbb{Z}^n} \|f\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)}. \tag{4.31}$$

Collecting (4.29)–(4.31) and using Fubini's Theorem, we have

$$\begin{aligned}
&\|v_1...v_K\|_{\ell^{1,s}_\triangle \ell^1_\alpha(L^q_t L^p_x(\mathbb{R}\times Q_\alpha))} \\
&\lesssim \sum_{k=1}^K \sum_{r=-1}^\infty \sum_{j\leq r+C} 2^{sj} \sum_{\alpha\in\mathbb{Z}^n} \left\| \triangle_j (\triangle_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^K S_{r+1} v_i) \right\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)} \\
&\lesssim \sum_{k=1}^K \sum_{r=-1}^\infty \sum_{j\leq r+C} 2^{sj} \sum_{\alpha\in\mathbb{Z}^n} \left\| \triangle_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^K S_{r+1} v_i \right\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)} \\
&\lesssim \sum_{k=1}^K \sum_{r=-1}^\infty 2^{sr} \sum_{\alpha\in\mathbb{Z}^n} \left\| \triangle_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^K S_{r+1} v_i \right\|_{L^q_t L^p_x(\mathbb{R}\times Q_\alpha)} \\
&\lesssim \sum_{k=1}^K \sum_{r=-1}^\infty 2^{sr} \sum_{\alpha\in\mathbb{Z}^n} \|\triangle_{r+1} v_k\|_{L^{q_1}_t L^{p_1}_x(\mathbb{R}\times Q_\alpha)} \prod_{i\neq k,\ i=1,...,K} \|v_i\|_{\ell^1_\triangle(L^{q_2}_t L^{p_2}_x(\mathbb{R}\times Q_\alpha))} \\
&\lesssim \sum_{k=1}^K \|v_k\|_{\ell^{1,s}_\triangle \ell^\infty_\alpha(L^{q_1}_t L^{p_1}_x(\mathbb{R}\times Q_\alpha))} \sum_{\alpha\in\mathbb{Z}^n} \prod_{i\neq k,\ i=1,...,K} \|v_i\|_{\ell^1_\triangle(L^{q_2}_t L^{p_2}_x(\mathbb{R}\times Q_\alpha))},
\end{aligned} \tag{4.32}$$

the result follows. □

For short, we write $\|\nabla u\|_X = \|\partial_{x_1} u\|_X + ... + \|\partial_{x_n} u\|_X$.

Lemma 4.3 *Let $n \geq 3$. We have for any $s > 0$*

$$\sum_{k=0,1} \|S(t)\nabla^k u_0\|_{\ell^{1,s}_\triangle \ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \lesssim \|u_0\|_{B^{s+1/2}_{2,1}}, \tag{4.33}$$
$$\sum_{k=0,1} \|\mathscr{A}\nabla^k F\|_{\ell^{1,s}_\triangle \ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \lesssim \|F\|_{\ell^{1,s}_\triangle \ell^1_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}. \tag{4.34}$$

Proof. In view of Corollary 3.5 and Propositions 3.1 and 3.6, we have the results, as desired. □

Lemma 4.4 *Let $n = 2$. We have for any $s > 0$*

$$\sum_{k=0,1} \|S(t)\nabla^k u_0\|_{\ell^{1,s}_\triangle \ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \lesssim \|u_0\|_{B^{s+1/2}_{2,1}\cap \dot H^{-1/2}}, \tag{4.35}$$
$$\|\mathscr{A}\nabla F\|_{\ell^{1,s}_\triangle \ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \lesssim \|F\|_{\ell^{1,s}_\triangle \ell^1_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}, \tag{4.36}$$
$$\|\mathscr{A} F\|_{\ell^{1,s}_\triangle \ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \lesssim \|F\|_{\ell^{1,s}_\triangle \ell^1_\alpha(L^1_t L^2_x(\mathbb{R}\times Q_\alpha))}. \tag{4.37}$$

Proof. By Propositions 3.2, 3.3 and 3.6, we have the results, as desired. □

We now continue the proof of Theorem 1.1 and now we consider the general case $s > n/2 + 3/2$. We write

$$\begin{aligned}
\lambda_1(v) &:= \sum_{i=0,1} \|\nabla^i v\|_{\ell^{1,s-1/2}_{\triangle}\ell^{\infty}_{\alpha}(L^2_{t,x}(\mathbb{R}\times Q_\alpha))},\\
\lambda_2(v) &:= \sum_{i=0,1} \|\nabla^i v\|_{\ell^{1,s-1/2}_{\triangle}\ell^{2^*}_{\alpha}(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))},\\
\lambda_3(v) &:= \sum_{i=0,1} \|\nabla^i v\|_{\ell^{1,s-1/2}_{\triangle}\ell^{2^*}_{\alpha}(L^{2m}_{t}L^\infty_x(\mathbb{R}\times Q_\alpha))},\\
\mathscr{D} &= \{u : \sum_{i=1,2,3} \lambda_i(v) \le \varrho\}. \qquad (4.38)
\end{aligned}$$

Note λ_i and $\mathscr{D}$ defined here are different from those in the above. We only give the details of the proof in the case $n \ge 3$ and the case $n = 2$ can be shown in a slightly different way. Let $\mathscr{T}$ be defined as in (4.4). Using Lemma 4.3, we have

$$\lambda_1(\mathscr{T}u) \lesssim \|u_0\|_{B^s_{2,1}} + \|F\|_{\ell^{1,s-1/2}_{\triangle}\ell^1_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}. \qquad (4.39)$$

For simplicity, we write

$$(u)^\kappa(\nabla u)^\nu = u^{\kappa_1}\bar{u}^{\kappa_2}u^{\nu_1}_{x_1}\bar{u}^{\nu_2}_{x_1}...u^{\nu_{2n-1}}_{x_n}\bar{u}^{\nu_{2n}}_{x_n}, \qquad (4.40)$$

$|\kappa| = \kappa_1 + \kappa_2$, $|\nu| = \nu_1 + ... + \nu_{2n}$. By Lemma 4.2, we have

$$\begin{aligned}
&\|(u)^\kappa(\nabla u)^\nu\|_{\ell^{1,s-1/2}_{\triangle}\ell^1_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}\\
&\lesssim \sum_{i=0,1} \|\nabla^i u\|_{\ell^{1,s-1/2}_{\triangle}\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))} \sum_{k=0,1} \|\nabla^k u\|^{|\kappa|+|\nu|-1}_{\ell^1_\triangle\ell^{|\kappa|+|\nu|-1}_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))}. \qquad (4.41)
\end{aligned}$$

Hence, if $u \in \mathscr{D}$, in view of (4.39) and (4.41), we have

$$\lambda_1(\mathscr{T}u) \lesssim \|u_0\|_{B^s_{2,1}} + \sum_{m+1\le|\kappa|+|\nu|\le M+1} \varrho^{|\kappa|+|\nu|}. \qquad (4.42)$$

In view of the estimate for the maximal function as in Proposition 2.5, one has that

$$\lambda_2(S(t)u_0) \lesssim \|u_0\|_{B^s_{2,1}}. \qquad (4.43)$$

and for $i = 0, 1$,

$$\begin{aligned}
\|\mathscr{A}\nabla^i F\|_{\ell^1_\triangle\ell^{2^*}_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))} &\le \sum_{j=0}^\infty \int_{\mathbb{R}} \|S(t-\tau)(\triangle_j\nabla^i F)(\tau)\|_{\ell^{2^*}_\alpha(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))}\\
&\lesssim \sum_{j=0}^\infty \int_{\mathbb{R}} \|(\triangle_j\nabla^i F)(\tau)\|_{H^{s-3/2}(\mathbb{R}^n)}d\tau\\
&\lesssim \sum_{j=0}^\infty 2^{(s-1/2)j} \int_{\mathbb{R}} \|(\triangle_j F)(\tau)\|_{L^2(\mathbb{R}^n)}d\tau. \qquad (4.44)
\end{aligned}$$

Hence, by (4.43) and (4.44),

$$\lambda_2(\mathscr{T}u) \lesssim \|u_0\|_{B^s_{2,1}} + \|F\|_{\ell^{1,s-1/2}_{\triangle}\ell^1_\alpha(L^1_t L^2_x(\mathbb{R}\times Q_\alpha))}. \tag{4.45}$$

Similar to (4.45), in view of Proposition 2.9, we have

$$\lambda_3(\mathscr{T}u) \lesssim \|u_0\|_{B^s_{2,1}} + \|F\|_{\ell^{1,s-1/2}_{\triangle}\ell^1_\alpha(L^1_t L^2_x(\mathbb{R}\times Q_\alpha))}. \tag{4.46}$$

In view of Lemma 4.2, we have

$$\begin{aligned}\|(u)^\kappa(\nabla u)^\nu\|_{\ell^{1,s-1/2}_{\triangle}\ell^1_\alpha(L^1_t L^2_x(\mathbb{R}\times Q_\alpha))} \lesssim & \sum_{k=0,1} \|\nabla^k u\|^{|\kappa|+|\nu|-1}_{\ell^1_{\triangle}\ell^{(|\kappa|+|\nu|-1)}_\alpha(L^{2(|\kappa|+|\nu|-1)}_t L^\infty_x(\mathbb{R}\times Q_\alpha))} \\ & \times \sum_{i=0,1} \|\nabla^i u\|_{\ell^{1,s-1/2}_{\triangle}\ell^\infty_\alpha(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}. \end{aligned} \tag{4.47}$$

Hence, if $u \in \mathscr{D}$, we have

$$\lambda_2(\mathscr{T}u) + \lambda_3(\mathscr{T}u) \lesssim \|u_0\|_{B^s_{2,1}} + \sum_{m+1\le|\kappa|+|\nu|\le M+1} \varrho^{|\kappa|+|\nu|}. \tag{4.48}$$

Repeating the procedures as in the above, we obtain that there exists $u \in \mathscr{D}$ satisfying the integral equation $\mathscr{T}u = u$, which finishes the proof of Theorem 1.1. $\square$

5 Proof of Theorem 1.2

We begin with the following

Lemma 5.1 *Let $\mathscr{A}$ be as in* (1.9). *There exists a constant $C(T) > 1$ which depends only on T and n such that*

$$\sum_{i=0,1} \|\mathscr{A}\nabla^i F\|_{\ell^1_\Box \ell^2_\alpha(L^\infty_{t,x}([0,T]\times Q_\alpha))} \le C(T)\|F\|_{\ell^{1,3/2}_\Box \ell^1_\alpha(L^1_t L^2_x([0,T]\times Q_\alpha))}. \tag{5.1}$$

Proof. Using Minkowski's inequality and Proposition 2.1,

$$\begin{aligned} & \|\mathscr{A}\nabla^i F\|_{\ell^1_\Box \ell^2_\alpha(L^\infty_{t,x}([0,T]\times Q_\alpha))} \\ & \le \sum_{k\in\mathbb{Z}^n} \left(\sum_{\alpha\in\mathbb{Z}^n} \left(\int_0^T \|S(t-\tau)\Box_k \nabla^i F(\tau)\|_{L^\infty_{t,x}([0,T]\times Q_\alpha)} d\tau \right)^2 \right)^{1/2} \\ & \le \sum_{k\in\mathbb{Z}^n} \int_0^T \left(\sum_{\alpha\in\mathbb{Z}^n} \|S(t-\tau)\Box_k \nabla^i F(\tau)\|^2_{L^\infty_{t,x}([0,T]\times Q_\alpha)} \right)^{1/2} d\tau \\ & \le \sum_{k\in\mathbb{Z}^n} \int_0^T \|\Box_k \nabla^i F(\tau)\|_{M^{1/2}_{2,1}} d\tau. \end{aligned} \tag{5.2}$$

It is easy to see that for $i=0,1$,

$$\|\Box_k \nabla^i F\|_{M^{1/2}_{2,1}} \lesssim \langle k\rangle^{3/2}\|\Box_k F\|_{L^2(\mathbb{R}^n)} \le \langle k\rangle^{3/2} \sum_{\alpha\in\mathbb{Z}^n} \|\Box_k F\|_{L^2(Q_\alpha)}. \tag{5.3}$$

By (5.2) and (5.3), we immediately have (5.1). $\Box$

Lemma 5.2 *Let $\mathscr{A}$ be as in* (1.9). *Let $n\ge 2$, $s>0$. Then we have*

$$\sum_{i=0,1} \|\nabla^i \mathscr{A} F\|_{\ell^{1,s}_\Box \ell^\infty_\alpha (L^2_{t,x}([0,T]\times Q_\alpha))} \le \langle T\rangle^{1/2} \|F\|_{\ell^{1,s}_\Box \ell^1_\alpha (L^2_{t,x}([0,T]\times Q_\alpha))}. \tag{5.4}$$

Proof. In view of Proposition 3.6, we have

$$\|\nabla \mathscr{A} F\|_{\ell^{1,s}_\Box \ell^\infty_\alpha (L^2_{t,x}([0,T]\times Q_\alpha))} \lesssim \|F\|_{\ell^{1,s}_\Box \ell^1_\alpha (L^2_{t,x}([0,T]\times Q_\alpha))}. \tag{5.5}$$

By Propositions 3.1 and 3.3,

$$\begin{aligned}\|\mathscr{A} F\|_{\ell^1_\Box \ell^\infty_\alpha (L^2_{t,x}([0,T]\times Q_\alpha))} &\lesssim \|F\|_{\ell^{1,s}_\Box \ell^1_\alpha (L^1_t L^2_x([0,T]\times Q_\alpha))} \\ &\le T^{1/2} \|F\|_{\ell^{1,s}_\Box \ell^1_\alpha (L^2_{t,x}([0,T]\times Q_\alpha))}. \end{aligned}\tag{5.6}$$

By (5.5) and (5.6) we immediately have (5.4). $\Box$

Lemma 5.3 *Let $n\ge 2$, $S(t)$ be as in* (1.9). *Then we have for $i=0,1$,*

$$\|\nabla^i S(t) u_0\|_{\ell^{1,s}_\Box \ell^\infty_\alpha (L^2_{t,x}([0,T]\times Q_\alpha))} \lesssim \|u_0\|_{M^{s+1/2}_{2,1}}, \quad n\ge 3, \tag{5.7}$$

$$\|\nabla^i S(t) u_0\|_{\ell^{1,s}_\Box \ell^\infty_\alpha (L^2_{t,x}([0,T]\times Q_\alpha))} \lesssim \|u_0\|_{M^{s+1/2}_{2,1}\cap \dot H^{-1/2}}, \quad n=2. \tag{5.8}$$

Proof. (5.7) follows from Corollary 3.5. For $n=2$, by Proposition 3.2, we have the result, as desired. $\Box$

Lemma 5.4 *Let $n\ge 2$, $s>0$, $L\in\mathbb{N}, L\ge 3$. Let $1\le p,p_i,q,q_i\le\infty$ satisfy $1/p=1/p_1+(L-1)/p_2$ and $1/q=1/q_1+(L-1)/q_2$. We have*

$$\begin{aligned}\|v_1\dots v_L\|_{\ell^{1,s}_\Box \ell^1_\alpha (L^q_t L^p_x(I\times Q_\alpha))} \lesssim & \sum_{l=1}^L \|v_l\|_{\ell^{1,s}_\Box \ell^\infty_\alpha (L^{q_1}_t L^{p_1}_x(I\times Q_\alpha))} \\ & \times \prod_{i\ne l,\, i=1,\dots,L} \|v_i\|_{\ell^1_\Box \ell^{L-1}_\alpha (L^{q_2}_t L^{p_2}_x(I\times Q_\alpha))}. \end{aligned}\tag{5.9}$$

Proof. Using the identity

$$v_1\dots v_L = \sum_{k_1,\dots,k_L\in\mathbb{Z}^n} \Box_{k_1} v_1 \dots \Box_{k_L} v_L \tag{5.10}$$

and noticing the fact that

$$\Box_k(\Box_{k_1}v_1...\Box_{k_L}v_L)=0,\quad |k-k_1-...-k_L|\geq C(L,n), \tag{5.11}$$

we have

$$\begin{aligned}
&\|v_1...v_L\|_{\ell_\Box^{1,s}\ell_\alpha^1(L_t^qL_x^p(I\times Q_\alpha))}\\
&=\sum_{k\in\mathbb{Z}^n}\langle k\rangle^s\sum_{\alpha\in\mathbb{Z}^n}\|\Box_k(v_1...v_L)\|_{L_t^qL_x^p(I\times Q_\alpha)}\\
&\leq\sum_{k_1,...,k_n\in\mathbb{Z}^n}\sum_{|k-k_1-...-k_L|\leq C}\langle k\rangle^s\sum_{\alpha\in\mathbb{Z}^n}\|\Box_k(\Box_{k_1}v_1...\Box_{k_L}v_L)\|_{L_t^qL_x^p(I\times Q_\alpha)}.
\end{aligned} \tag{5.12}$$

Similar to (4.31) and noticing the fact that $\|\mathscr{F}^{-1}\sigma_k\|_{L^1(\mathbb{R}^n)}\lesssim 1$, we have

$$\begin{aligned}
\sum_{\alpha\in\mathbb{Z}^n}\|\,\Box_kf\|_{L_t^qL_x^p(I\times Q_\alpha)}&=\sum_{\alpha\in\mathbb{Z}^n}\|(\mathscr{F}^{-1}\sigma_k)*f\|_{L_t^qL_x^p(I\times Q_\alpha)}\\
&\leq\int_{\mathbb{R}^n}|(\mathscr{F}^{-1}\sigma_k)(y)|\left(\sum_{\alpha\in\mathbb{Z}^n}\|f(t,x-y)\|_{L_t^qL_x^p(I\times Q_\alpha)}\right)dy\\
&\leq\sup_{y\in\mathbb{R}^n}\sum_{\alpha\in\mathbb{Z}^n}\|f(t,x-y)\|_{L_t^qL_x^p(I\times Q_\alpha)}\|\mathscr{F}^{-1}\sigma_k\|_{L^1(\mathbb{R}^n)}\\
&\lesssim\sum_{\alpha\in\mathbb{Z}^n}\|f\|_{L_t^qL_x^p(I\times Q_\alpha)}.
\end{aligned} \tag{5.13}$$

By (5.12) and (5.13), we have

$$\begin{aligned}
&\|v_1...v_L\|_{\ell_\Box^{1,s}\ell_\alpha^1(L_t^qL_x^p(I\times Q_\alpha))}\\
&\leq\sum_{k_1,...,k_n\in\mathbb{Z}^n}\sum_{|k-k_1-...-k_L|\leq C}\langle k\rangle^s\sum_{\alpha\in\mathbb{Z}^n}\|\Box_{k_1}v_1...\Box_{k_L}v_L\|_{L_t^qL_x^p(I\times Q_\alpha)}\\
&\lesssim\sum_{k_1,...,k_n\in\mathbb{Z}^n}(\langle k_1\rangle^s+...+\langle k_L\rangle^s)\sum_{\alpha\in\mathbb{Z}^n}\|\Box_{k_1}v_1...\Box_{k_L}v_L\|_{L_t^qL_x^p(I\times Q_\alpha)}.
\end{aligned} \tag{5.14}$$

By Hölder's inequality,

$$\begin{aligned}
&\sum_{k_1,...,k_n\in\mathbb{Z}^n}\langle k_1\rangle^s\sum_{\alpha\in\mathbb{Z}^n}\|\Box_{k_1}v_1...\Box_{k_L}v_L\|_{L_t^qL_x^p(I\times Q_\alpha)}\\
&\leq\sum_{k_1,...,k_n\in\mathbb{Z}^n}\langle k_1\rangle^s\sum_{\alpha\in\mathbb{Z}^n}\|\Box_{k_1}v_1\|_{L_t^{q_1}L_x^{p_2}(I\times Q_\alpha)}\prod_{i=2}^L\|\Box_{k_i}v_i\|_{L_t^{q_2}L_x^{p_2}(I\times Q_\alpha)}\\
&\leq\|v_1\|_{\ell_\Box^{1,s}\ell_\alpha^\infty(L_t^{q_1}L_x^{p_2}(I\times Q_\alpha))}\sum_{k_2,...,k_n\in\mathbb{Z}^n}\sum_{\alpha\in\mathbb{Z}^n}\prod_{i=2}^L\|\Box_{k_i}v_i\|_{L_t^{q_2}L_x^{p_2}(I\times Q_\alpha)}\\
&\leq\|v_1\|_{\ell_\Box^{1,s}\ell_\alpha^\infty(L_t^{q_1}L_x^{p_2}(I\times Q_\alpha))}\sum_{k_2,...,k_n\in\mathbb{Z}^n}\prod_{i=2}^L\|\Box_{k_i}v_i\|_{\ell_\alpha^{L-1}(L_t^{q_2}L_x^{p_2}(I\times Q_\alpha))}\\
&\leq\|v_1\|_{\ell_\Box^{1,s}\ell_\alpha^\infty(L_t^{q_1}L_x^{p_2}(I\times Q_\alpha))}\prod_{i=2}^L\|v_i\|_{\ell_\Box^1\ell_\alpha^{L-1}(L_t^{q_2}L_x^{p_2}(I\times Q_\alpha))}.
\end{aligned} \tag{5.15}$$

The result follows. $\Box$

Proof of Theorem 1.2. Denote

$$\lambda_1(v) = \sum_{i=0,1} \|\nabla^i v\|_{\ell_\square^{1,3/2}\ell_\alpha^\infty(L^2_{t,x}([0,T]\times Q_\alpha))},$$
$$\lambda_2(v) = \sum_{i=0,1} \|\nabla^i v\|_{\ell_\square^{1}\ell_\alpha^2(L^\infty_{t,x}([0,T]\times Q_\alpha))}.$$

Put

$$\mathscr{D} = \{u : \ \lambda_1(u) + \lambda_2(u) \le \varrho\}. \tag{5.16}$$

Let $\mathscr{T}$ be as in (4.4). We will show that $\mathscr{T} : \mathscr{D} \to \mathscr{D}$ is a contraction mapping. First, we consider the case $n \ge 3$. Let $u \in \mathscr{D}$. By Lemmas 5.2 and 5.3, we have

$$\lambda_1(\mathscr{T}u) \lesssim \|u_0\|_{M^2_{2,1}} + \langle T\rangle^{1/2} \|F\|_{\ell_\square^{1,3/2}\ell_\alpha^1(L^2_{t,x}([0,T]\times Q_\alpha))}. \tag{5.17}$$

We use the same notation as in (4.40). We have from Lemma 5.4 that

$$\begin{aligned} \|(u)^\kappa(\nabla u)^\nu\|_{\ell_\square^{1,3/2}\ell_\alpha^1(L^2_{t,x}([0,T]\times Q_\alpha))} &\lesssim \sum_{i=0,1} \|\nabla^i u\|_{\ell_\square^{1,3/2}\ell_\alpha^\infty(L^2_{t,x}([0,T]\times Q_\alpha))} \\ &\quad \times \sum_{k=0,1} \|\nabla^k u\|^{|\kappa|+|\nu|-1}_{\ell_\square^1\ell_\alpha^{|\kappa|+|\nu|-1}(L^\infty_{t,x}([0,T]\times Q_\alpha))} \\ &\lesssim \lambda_1(u)\lambda_2(u)^{|\kappa|+|\nu|-1} \le \varrho^{|\kappa|+|\nu|}. \end{aligned} \tag{5.18}$$

Hence, for $n \ge 3$,

$$\lambda_1(\mathscr{T}u) \lesssim \|u_0\|_{M^2_{2,1}} + \sum_{|\kappa|+|\nu|=m+1}^{M} \varrho^{|\kappa|+|\nu|}. \tag{5.19}$$

Next, we consider the estimate of $\lambda_2(\mathscr{T}u)$. By Lemma 5.1 and Proposition 2.1,

$$\lambda_2(\mathscr{T}u) \lesssim \|u_0\|_{M^{3/2}_{2,1}} + C(T)\|F\|_{\ell_\square^{1,3/2}\ell_\alpha^1(L^1_t L^2_x([0,T]\times Q_\alpha))}, \tag{5.20}$$

which reduces to the estimates of $\lambda_1(\cdot)$ as in (5.17). Similarly, for $n = 2$,

$$\lambda_1(\mathscr{T}u) + \lambda_2(\mathscr{T}u) \lesssim \|u_0\|_{M^2_{2,1}\cap \dot H^{-1/2}} + \sum_{|\kappa|+|\nu|=m+1}^{M} \varrho^{|\kappa|+|\nu|}. \tag{5.21}$$

Repeating the procedures as in the proof of Theorem 1.1, we can show our results, as desired. $\square$

6 Proof of Theorem 1.3

The proof of Theorem 1.3 follows an analogous way as that in Theorems 1.1 and 1.2 and will be sketched. Put

$$\begin{aligned}
\lambda_1(v) &= \sum_{i=0,1} \|\nabla^i v\|_{\ell_\square^{1,s-1/2}\ell_\alpha^\infty(L^2_{t,x}(\mathbb{R}\times Q_\alpha))},\\
\lambda_2(v) &= \sum_{i=0,1} \|\nabla^i v\|_{\ell_\square^{1}\ell_\alpha^m(L^\infty_{t,x}(\mathbb{R}\times Q_\alpha))},\\
\lambda_3(v) &= \sum_{i=0,1} \|\nabla^i v\|_{\ell_\square^{1}\ell_\alpha^m(L^{2m}_{t}L^\infty_x(\mathbb{R}\times Q_\alpha))}.
\end{aligned}$$

Put

$$\mathscr{D} = \{u : \ \lambda_1(u) + \lambda_2(u) + \lambda_3(u) \le \varrho\}. \tag{6.1}$$

Let $\mathscr{T}$ be as in (4.4). We show that $\mathscr{T} : \mathscr{D} \to \mathscr{D}$. We only consider the case $n \ge 3$. It follows from Lemma 5.3 and 4.3 that

$$\lambda_1(\mathscr{T}u) \lesssim \|u_0\|_{M^s_{2,1}} + \|F\|_{\ell_\square^{1,s-1/2}\ell_\alpha^1(L^2_{t,x}(\mathbb{R}\times Q_\alpha))}. \tag{6.2}$$

Using Lemma 5.4 and similar to (5.18), one sees that if $u \in \mathscr{D}$, then

$$\lambda_1(\mathscr{T}u) \lesssim \|u_0\|_{M^s_{2,1}} + \sum_{m+1\le|\kappa|+|\nu|\le M+1} \varrho^{|\kappa|+|\nu|}. \tag{6.3}$$

Using Proposition 2.10 and combining the proof of (4.44)–(4.46), we see that

$$\lambda_2(\mathscr{T}u) + \lambda_3(\mathscr{T}u) \lesssim \|u_0\|_{M^s_{2,1}} + \sum_{m+1\le|\kappa|+|\nu|\le M+1} \varrho^{|\kappa|+|\nu|}. \tag{6.4}$$

The left part of the proof is analogous to that of Theorems 1.1 and 1.2 and the details are omitted. $\square$

7 Proof of Theorem 1.4

We prove Theorem 1.4 by following some ideas as in Molinet and Ribaud [23] and Wang and Huang [33]. The following is the estimates for the solutions of the linear Schrödinger equation, see [16,23,33]. Recall that $\triangle_j := \mathscr{F}^{-1}\delta(2^{-j}\,\cdot)\mathscr{F}$, $j \in \mathbb{Z}$ and $\delta(\cdot)$ is as in Section 1.4.

Lemma 7.1 *Let $g \in \mathscr{S}(\mathbb{R})$, $f \in \mathscr{S}(\mathbb{R}^2)$, $4 \le p < \infty$. Then we have*

$$\|\triangle_j S(t) g\|_{L^\infty_t L^2_x \cap L^6_{x,t}} \lesssim \|\triangle_j g\|_{L^2}, \tag{7.1}$$

$$\|\triangle_j S(t)g\|_{L^p_x L^\infty_t} \lesssim 2^{j(\frac{1}{2}-\frac{1}{p})}\|\triangle_j g\|_{L^2}, \tag{7.2}$$
$$\|\triangle_j S(t)g\|_{L^\infty_x L^2_t} \lesssim 2^{-j/2}\|\triangle_j g\|_{L^2}, \tag{7.3}$$

$$\|\triangle_j \mathscr{A} f\|_{L^\infty_t L^2_x \cap L^6_{x,t}} \lesssim \|\triangle_j f\|_{L^{6/5}_{x,t}}, \tag{7.4}$$
$$\|\triangle_j \mathscr{A} f\|_{L^p_x L^\infty_t} \lesssim 2^{j(\frac{1}{2}-\frac{1}{p})}\|\triangle_j f\|_{L^{6/5}_{x,t}}, \tag{7.5}$$
$$\|\triangle_j \mathscr{A} f\|_{L^\infty_x L^2_t} \lesssim 2^{-j/2}\|\triangle_j f\|_{L^{6/5}_{x,t}}, \tag{7.6}$$

and

$$\|\triangle_j \mathscr{A}(\partial_x f)\|_{L^\infty_t L^2_x \cap L^6_{x,t}} \lesssim 2^{j/2}\|\triangle_j f\|_{L^1_x L^2_t}, \tag{7.7}$$
$$\|\triangle_j \mathscr{A}(\partial_x f)\|_{L^p_x L^\infty_t} \lesssim 2^{j/2} 2^{j(\frac{1}{2}-\frac{1}{p})}\|\triangle_j f\|_{L^1_x L^2_t}, \tag{7.8}$$
$$\|\triangle_j \mathscr{A}(\partial_x f)\|_{L^\infty_x L^2_t} \lesssim \|\triangle_j f\|_{L^1_x L^2_t}. \tag{7.9}$$

For convenience, we write for any Banach function space X,

$$\|f\|_{\ell^{1,s}_\triangle(X)} = \sum_{j\in\mathbb{Z}} 2^{js}\|\triangle_j f\|_X, \quad \|f\|_{\ell^1_\triangle(X)} := \|f\|_{\ell^{1,0}_\triangle(X)}.$$

Lemma 7.2 *Let $s > 0$, $1 \le p, p_i, \gamma, \gamma_i \le \infty$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_N}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_N}. \tag{7.10}$$

Then

$$\begin{aligned} \|u_1 \dots u_N\|_{\ell^{1,s}_\triangle(L^p_x L^\gamma_t)} \lesssim{}& \|u_1\|_{\ell^{1,s}_\triangle(L^{p_1}_x L^{\gamma_1}_t)} \prod_{i=2}^N \|u_i\|_{\ell^1_\triangle(L^{p_i}_x L^{\gamma_i}_t)} \\ &+ \|u_2\|_{\ell^{1,s}_\triangle(L^{p_2}_x L^{\gamma_2}_t)} \prod_{i\neq 2,\, i=1,\dots,N} \|u_i\|_{\ell^1_\triangle(L^{p_i}_x L^{\gamma_i}_t)} \\ &+ \dots + \prod_{i=1}^{N-1} \|u_i\|_{\ell^1_\triangle(L^{p_i}_x L^{\gamma_i}_t)} \|u_N\|_{\ell^{1,s}_\triangle(L^{p_N}_x L^{\gamma_N}_t)}, \end{aligned} \tag{7.11}$$

and in particular, if $u_1 = \dots = u_N = u$, then

$$\left\|u^N\right\|_{\ell^{1,s}_\triangle(L^p_x L^\gamma_t)} \lesssim \|u\|_{\ell^{1,s}_\triangle(L^{p_1}_x L^{\gamma_1}_t)} \prod_{i=2}^N \|u\|_{\ell^1_\triangle(L^{p_i}_x L^{\gamma_i}_t)}. \tag{7.12}$$

Substituting the spaces $L^p_x L^\gamma_t$ and $L^{p_i}_x L^{\gamma_i}_t$ by $L^\gamma_t L^p_x$ and $L^{\gamma_i}_t L^{p_i}_x$, respectively, (7.11) *and* (7.12) *also holds.*

Proof. We only consider the case $N=2$ and the case $N>2$ can be handled in a similar way. We have

$$\begin{aligned} u_1u_2 &= \sum_{r=-\infty}^{\infty}[(S_{r+1}u_1)(S_{r+1}u_2)-(S_ru_1)(S_ru_2)] \\ &= \sum_{r=-\infty}^{\infty}[(\triangle_{r+1}u_1)(S_{r+1}u_2)+(S_ru_1)(\triangle_{r+1}u_2)], \end{aligned} \tag{7.13}$$

and

$$\triangle_j(u_1u_2)=\triangle_j\Big(\sum_{r\geq j-10}[(\triangle_{r+1}u_1)(S_{r+1}u_2)+(S_ru_1)(\triangle_{r+1}u_2)]\Big). \tag{7.14}$$

We may assume, without loss of generality that there is only the first term in the right hand side of (7.14) and the second term can be handled in the same way. It follows from Bernstein's estimate, Hölder's and Young's inequalities that

$$\begin{aligned} \sum_{j\in\mathbb{Z}}2^{sj}\|\triangle_j(u_1u_2)\|_{L^p_xL^\gamma_t} &\lesssim \sum_{j\in\mathbb{Z}}2^{sj}\sum_{r\geq j-10}\|(\triangle_{r+1}u_1)(S_{r+1}u_2)\|_{L^p_xL^\gamma_t} \\ &\lesssim \sum_{j\in\mathbb{Z}}2^{sj}\sum_{r\geq j-10}\|\triangle_{r+1}u_1\|_{L^{p_1}_xL^{\gamma_1}_t}\|S_{r+1}u_2\|_{L^{p_2}_xL^{\gamma_2}_t} \\ &\lesssim \sum_{j\in\mathbb{Z}}2^{s(j-r)}\sum_{r\geq j-10}2^{rs}\|\triangle_{r+1}u_1\|_{L^{p_1}_xL^{\gamma_1}_t}\|S_{r+1}u_2\|_{L^{p_2}_xL^{\gamma_2}_t} \\ &\lesssim \|u_1\|_{\ell^{1,s}_\triangle(L^{p_1}_xL^{\gamma_1}_t)}\|u_2\|_{\ell^1_\triangle(L^{p_2}_xL^{\gamma_2}_t)}, \end{aligned} \tag{7.15}$$

which implies the result, as desired. □

Remark 7.3 One easily sees that (7.12) can be slightly improved by

$$\left\|u^N\right\|_{\ell^{1,s}(L^p_xL^\gamma_t)}\lesssim \|u\|_{\ell^{1,s}(L^{p_1}_xL^{\gamma_1}_t)}\prod_{i=2}^{N}\|u\|_{L^{p_i}_xL^{\gamma_i}_t}. \tag{7.16}$$

In fact, from Minkowski's inequality it follows that

$$\|S_ru\|_{L^p_xL^\gamma_t}\lesssim\|u\|_{L^p_xL^\gamma_t}. \tag{7.17}$$

From (7.15) and (7.17) we get (7.16).

Proof of Theorem 1.4. We can assume, without loss of generality that

$$F(u,\bar u,u_x,\bar u_x)=\sum_{m+1\leq\kappa+\nu\leq M+1}\lambda_{\kappa\nu}u^\kappa u_x^\nu \tag{7.18}$$

and the general case can be handled in the same way.

Step 1. We consider the case $m > 4$. Recall that

$$\|u\|_X = \sup_{s_m \le s \le \tilde{s}_M} \sum_{i=0,1} \sum_{j \in \mathbb{Z}} \|\!|\partial_x^i \triangle_j u\|\!|_s, \tag{7.19}$$

$$\begin{aligned} \|\!|\triangle_j v\|\!|_s := & 2^{sj} (\|\triangle_j v\|_{L_t^\infty L_x^2 \cap L^6_{x,t}} + 2^{j/2} \|\triangle_j v\|_{L_x^\infty L_t^2}) \\ & + 2^{(s-\tilde{s}_m)j} \|\triangle_j v\|_{L_x^m L_t^\infty} + 2^{(s-\tilde{s}_M)j} \|\triangle_j v\|_{L_x^M L_t^\infty}. \end{aligned} \tag{7.20}$$

Considering the mapping

$$\mathscr{T} : u(t) \to S(t)u_0 - \mathrm{i}\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x), \tag{7.21}$$

we will show that $\mathscr{T} : X \to X$ is a contraction mapping. We have

$$\|\mathscr{T} u(t)\|_X \lesssim \|S(t)u_0\|_X + \|\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_X. \tag{7.22}$$

In view of (7.1), (7.2) and (7.3) we have,

$$\|\!|\partial_x^i \triangle_j S(t) u_0\|\!|_s \lesssim 2^{sj} \|\partial_x^i \triangle_j u_0\|_2. \tag{7.23}$$

It follows that

$$\|S(t)u_0\|_X \lesssim \sup_{s_m \le s \le \tilde{s}_M} \sum_{i=0,1} \sum_{j \in \mathbb{Z}} 2^{sj} \|\partial_x^i \triangle_j u_0\|_2 \lesssim \|u_0\|_{\dot{B}^{s_m}_{2,1} \cap \dot{B}^{1+\tilde{s}_M}_{2,1}}. \tag{7.24}$$

We now estimate $\|\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_X$. We have from (7.4), (7.5) and (7.6) that

$$\|\!|\triangle_j (\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x))\|\!|_s \lesssim 2^{sj} \|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L^{6/5}_{x,t}}. \tag{7.25}$$

From (7.7), (7.8) and (7.9) it follows that

$$\|\!|\triangle_j (\mathscr{A} \partial_x F(u, \bar{u}, u_x, \bar{u}_x))\|\!|_s \lesssim 2^{sj} 2^{j/2} \|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L^1_x L^2_t}. \tag{7.26}$$

Hence, from (7.19), (7.25) and (7.26) we have

$$\begin{aligned} \|\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_X \lesssim & \sum_{j \in \mathbb{Z}} 2^{sj} \|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L^{6/5}_{x,t}} \\ & + \sum_{j \in \mathbb{Z}} 2^{sj} 2^{j/2} \|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L^1_x L^2_t} = I + II. \end{aligned} \tag{7.27}$$

Now we perform the nonlinear estimates. By Lemma 7.2,

$$\begin{aligned} I \lesssim & \sum_{m+1 \le \kappa+\nu \le M+1} \Big(\|u\|_{\ell^{1,s}_\triangle (L^6_{x,t})} \|u\|^{\kappa-1}_{\ell^1_\triangle (L^{3(\kappa+\nu-1)/2}_{x,t})} \|u_x\|^{\nu}_{\ell^1_\triangle (L^{3(\kappa+\nu-1)/2}_{x,t})} \\ & \quad + \|u_x\|_{\ell^{1,s}_\triangle (L^6_{x,t})} \|u_x\|^{\nu-1}_{\ell^1_\triangle (L^{3(\kappa+\nu-1)/2}_{x,t})} \|u\|^{\kappa}_{\ell^1_\triangle (L^{3(\kappa+\nu-1)/2}_{x,t})} \Big) \\ \lesssim & \sum_{m+1 \le \kappa+\nu \le M+1} \Big(\sum_{i=0,1} \|\partial_x^i u\|_{\ell^{1,s}_\triangle (L^6_{x,t})} \Big) \Big(\sum_{i=0,1} \|\partial_x^i u\|^{\kappa+\nu-1}_{\ell^1_\triangle (L^{3(\kappa+\nu-1)/2}_{x,t})} \Big). \end{aligned} \tag{7.28}$$

For any $m \le \lambda \le M$, we let $\frac{1}{\rho} = \frac{1}{2} - \frac{4}{3\lambda}$. It is easy to see that the following inclusions hold:

$$L_t^\infty(\mathbb{R}, \dot H^{s_\lambda}) \cap L_t^6(\mathbb{R}, \dot H_6^{s_\lambda}) \subset L_t^{3\lambda/2}(\mathbb{R}, \dot H_\rho^{s_\lambda}) \subset L_{x,t}^{3\lambda/2}. \tag{7.29}$$

More precisely, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \|\triangle_j u\|_{L_{x,t}^{3\lambda/2}} &\lesssim \sum_{j\in\mathbb{Z}} \|\triangle_j u\|_{L_t^{3\lambda/2}(\mathbb{R}, \dot H_\rho^{s_\lambda})} \\ &\lesssim \sum_{j\in\mathbb{Z}} \|\triangle_j u\|^{4/\lambda}_{L_t^6(\mathbb{R}, \dot H_6^{s_\lambda})} \|\triangle_j u\|^{1-4/\lambda}_{L_t^\infty(\mathbb{R}, \dot H^{s_\lambda})} \\ &\lesssim \|u\|^{4/\lambda}_{\ell^{1,s_\lambda}(L^6_{x,t})} \|u\|^{1-4/\lambda}_{\ell^{1,s_\lambda}(L_t^\infty L_x^2)}. \end{aligned} \tag{7.30}$$

Using (7.30) and noticing that $s_m \le s_{\kappa+\nu-1} \le s_M < \tilde s_M$, we have

$$\begin{aligned}\|\partial_x^i u\|^{\kappa+\nu-1}_{\ell^1_\triangle(L_{x,t}^{3(\kappa+\nu-1)/2})} &\lesssim \|\partial_x^i u\|^4_{\ell_\triangle^{1,\, s_{\kappa+\nu-1}}(L^6_{x,t})} \|\partial_x^i u\|^{\kappa+\nu-5}_{\ell_\triangle^{1,\, s_{\kappa+\nu-1}}(L_t^\infty L_x^2)} \\ &\lesssim \|u\|_X^{\kappa+\nu-1}. \end{aligned} \tag{7.31}$$

Combining (7.28) with (7.31), we have

$$I \lesssim \sum_{m+1 \le \kappa+\nu \le M+1} \|u\|_X^{\kappa+\nu}. \tag{7.32}$$

Now we estimate II. By Lemma 7.2,

$$\begin{aligned} II \lesssim & \sum_{m+1\le\kappa+\nu\le M+1} \Big(\|u\|_{\ell_\triangle^{1,\, s+1/2}(L_x^\infty L_t^2)} \|u\|^{\kappa-1}_{\ell^1_\triangle(L_x^{\kappa+\nu-1} L_t^\infty)} \|u_x\|^{\nu}_{\ell^1_\triangle(L_x^{\kappa+\nu-1} L_t^\infty)} \\ & \qquad + \|u_x\|_{\ell_\triangle^{1,\, s+1/2}(L_x^\infty L_t^2)} \|u_x\|^{\nu-1}_{L_x^{\kappa+\nu-1} L_t^\infty} \|u\|^{\kappa}_{L_x^{\kappa+\nu-1} L_t^\infty} \Big) \\ \lesssim & \sum_{m+1\le\kappa+\nu\le M+1} \Big(\sum_{i=0,1} \|\partial_x^i u\|_{\ell_\triangle^{1,\, s+1/2}(L_x^\infty L_t^2)} \Big) \Big(\sum_{i=0,1} \|\partial_x^i u\|^{\kappa+\nu-1}_{L_x^{\kappa+\nu-1} L_t^\infty} \Big) \\ \lesssim & \sum_{m+1\le\kappa+\nu\le M+1} \|u\|_X \Big(\sum_{i=0,1} \|\partial_x^i u\|^{\kappa+\nu-1}_{L_x^m L_t^\infty \cap L_x^M L_t^\infty} \Big) \\ \lesssim & \sum_{m+1\le\kappa+\nu\le M+1} \|u\|_X^{\kappa+\nu}. \end{aligned} \tag{7.33}$$

Collecting (7.27), (7.28), (7.32) and (7.33), we have

$$\|\mathscr{A} F(u, \bar u, u_x, \bar u_x)\|_X \lesssim \sum_{m+1\le\kappa+\nu\le M+1} \|u\|_X^{\kappa+\nu}. \tag{7.34}$$

By (7.22), (7.24) and (7.34)

$$\|\mathscr{T} u(t)\|_X \lesssim \|u_0\|_{\dot B^{s_m}_{2,1} \cap \dot B^{1+\tilde s_M}_{2,1}} + \sum_{m+1\le\kappa+\nu\le M+1} \|u\|_X^{\kappa+\nu}. \tag{7.35}$$

Step 2. We consider the case $m = 4$. Recall that

$$\|u\|_X = \sum_{i=0,1} \Big(\|\partial_x^i u\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} + \sup_{s_5 \le s \le \tilde{s}_M} \sum_{j \in \mathbb{Z}} \|\partial_x^i \triangle_j u\|_s \Big).$$

By (7.1), (7.2) and (7.3),

$$\|S(t)u_0\|_X \lesssim \|u_0\|_2 + \sup_{s_5 \le s \le \tilde{s}_M} \sum_{i=0,1} \sum_{j \in \mathbb{Z}} 2^{sj} \|\partial_x^i \triangle_j u_0\|_2 \lesssim \|u_0\|_{B_{2,1}^{1+\tilde{s}_M}}. \tag{7.36}$$

We now estimate $\|\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_X$. By Strichartz' and Hölder's inequality, we have

$$\begin{aligned}
&\|\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} \\
&\lesssim \sum_{5 \le \kappa+\nu \le M+1} \|(|u| + |u_x|)^{\kappa+\nu}\|_{L_{x,t}^{6/5}} \\
&\lesssim \sum_{5 \le \kappa+\nu \le M+1} \|(|u| + |u_x|)\|_{L_{x,t}^6} \|(|u| + |u_x|)\|^{\kappa+\nu-1}_{L_{x,t}^{3(\kappa+\nu-1)/2}} \\
&\lesssim \sum_{5 \le \kappa+\nu \le M+1} \|(|u| + |u_x|)\|_{L_{x,t}^6} \|(|u| + |u_x|)\|^{\kappa+\nu-1}_{L_{x,t}^6 \cap L_{x,t}^{3M/2}} \\
&\lesssim \sum_{5 \le \kappa+\nu \le M+1} \Big(\sum_{i=0,1} \|\partial_x^i u\|_{L_{x,t}^6} \Big)^{\kappa+\nu} \\
&\quad + \sum_{5 \le \kappa+\nu \le M+1} \Big(\sum_{i=0,1} \|\partial_x^i u\|_{L_{x,t}^6} \Big) \Big(\sum_{i=0,1} \|\partial_x^i u\|_{L_{x,t}^{3M/2}} \Big)^{\kappa+\nu-1}.
\end{aligned} \tag{7.37}$$

Applying (7.30), we see that (7.37) implies that

$$\|\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} \lesssim \sum_{5 \le \kappa+\nu \le M+1} \|u\|_X^{\kappa+\nu}. \tag{7.38}$$

From Bernstein's estimate and (7.7) it follows that

$$\begin{aligned}
&\|\partial_x \mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} \\
&\le \|P_{\le 1}(\mathscr{A} \partial_x F(u, \bar{u}, u_x, \bar{u}_x))\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} \\
&\quad + \|P_{>1}(\mathscr{A} \partial_x F(u, \bar{u}, u_x, \bar{u}_x))\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} \\
&\lesssim \|\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} \\
&\quad + \sum_{j \gtrsim 1} 2^{j/2} \|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_x^1 L_t^2} \\
&\lesssim \|\mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} \\
&\quad + \sum_{j \in \mathbb{Z}} 2^{\tilde{s}_M} 2^{j/2} \|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_x^1 L_t^2} = III + IV.
\end{aligned} \tag{7.39}$$

The estimates of III and IV have been given in (7.38) and (7.33), respectively. We have

$$\|\partial_x \mathscr{A} F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^\infty L_x^2 \cap L_{x,t}^6} \lesssim \sum_{5 \le \kappa+\nu \le M+1} \|u\|_X^{\kappa+\nu}. \tag{7.40}$$

We have from (7.4)–(7.6), (7.7)–(7.9) that

$$\sum_{j\in\mathbb{Z}} \|\triangle_j(\mathscr{A}F(u,\bar u,u_x,\bar u_x))\|_s \lesssim \sum_{j\in\mathbb{Z}} 2^{sj}\|\triangle_j F(u,\bar u,u_x,\bar u_x)\|_{L^{6/5}_{x,t}}, \tag{7.41}$$

$$\sum_{j\in\mathbb{Z}} \|\triangle_j(\mathscr{A}\partial_x F(u,\bar u,u_x,\bar u_x))\|_s \lesssim \sum_{j\in\mathbb{Z}} 2^{sj}2^{j/2}\|\triangle_j F(u,\bar u,u_x,\bar u_x)\|_{L^1_x L^2_t} \tag{7.42}$$

hold for all $s > 0$. The right hand side in (7.42) has been estimated by (7.33). So, it suffices to consider the estimate of the right hand side in (7.41). Let us observe the equality

$$F(u,\bar u,u_x,\bar u_x) = \sum_{\kappa+\nu=5} \lambda_{\kappa\nu} u^\kappa u_x^\nu + \sum_{5<\kappa+\nu\le M+1} \lambda_{\kappa\nu} u^\kappa u_x^\nu := V + VI. \tag{7.43}$$

For any $s_5 \le s \le \tilde s_M$, VI has been handled in (7.28)–(7.32):

$$\sum_{5<\kappa+\nu\le M+1}\sum_{j\in\mathbb{Z}} 2^{sj}\|\triangle_j(u^\kappa u_x^\nu)\|_{L^{6/5}_{x,t}} \lesssim \sum_{5<\kappa+\nu\le M+1} \|u\|_X^{\kappa+\nu}. \tag{7.44}$$

For the estimate of V, we use Remark 7.3, for any $s_5 \le s \le \tilde s_M$,

$$\begin{aligned}\sum_{\kappa+\nu=5}\sum_{j\in\mathbb{Z}} 2^{sj}\|\triangle_j(u^\kappa u_x^\nu)\|_{L^{6/5}_{x,t}} &\lesssim \Big(\sum_{i=0,1}\|\partial_x^i u_x\|^4_{L^6_{x,t}}\Big)\Big(\sum_{i=0,1}\|\partial_x^i u_x\|_{\ell^{1,s}_\triangle(L^6_{x,t})}\Big)\\ &\lesssim \|u\|_X^5. \end{aligned}\tag{7.45}$$

Summarizing the estimate above,

$$\|\mathscr{T}u(t)\|_X \lesssim \|u_0\|_{B^{1+\tilde s_M}_{2,1}} + \sum_{5\le\kappa+\nu\le M+1} \|u\|_X^{\kappa+\nu}, \tag{7.46}$$

whence, we have the results, as desired. $\square$

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